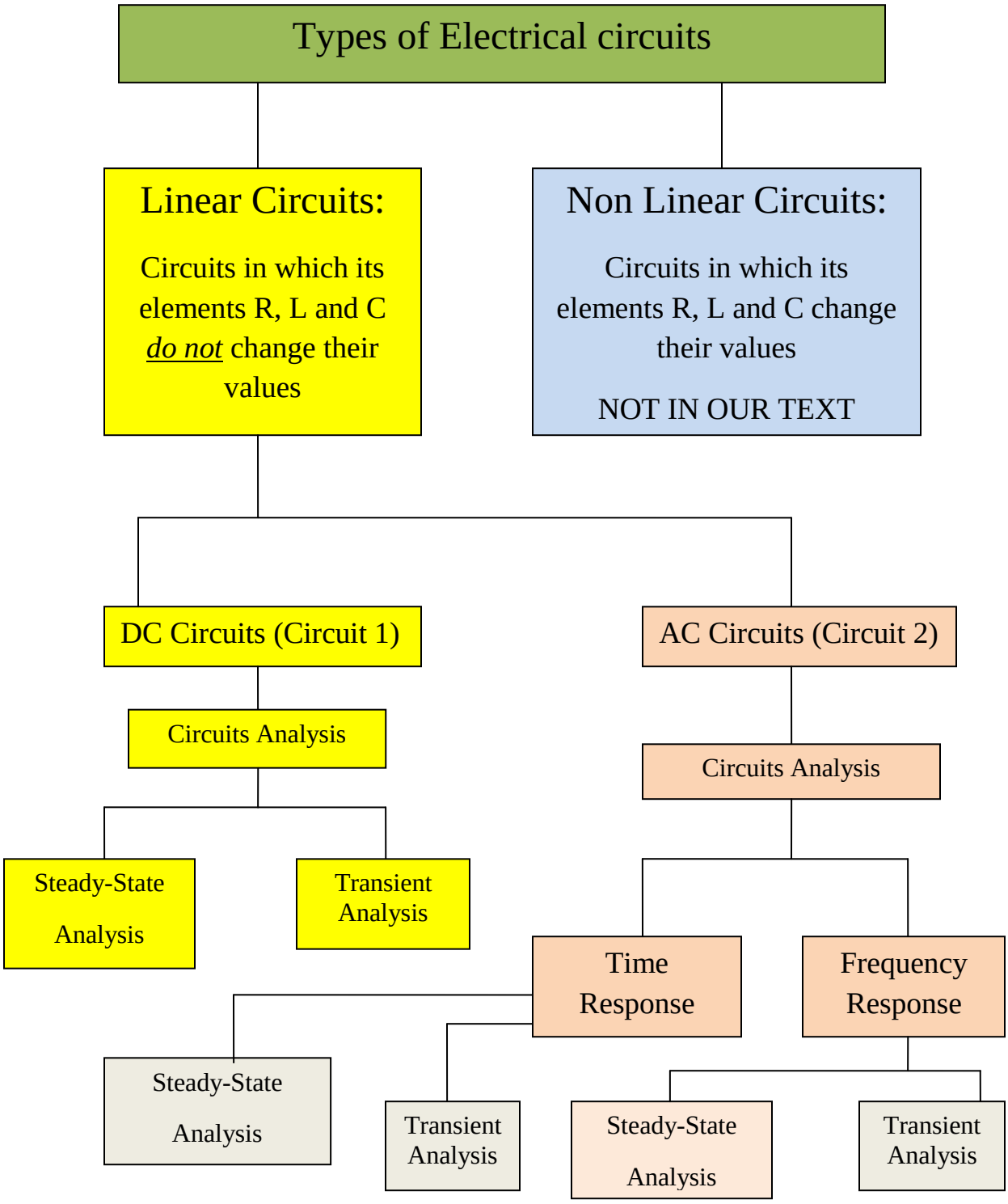
Electrical Circuit: An **electrical circuit** is a mathematical model that approximates the behavior of an actual electrical system.



Alternating Current (AC) Circuits Theory

1. INTRODUCTION

The analysis in circuit 1 has been limited to dc networks, networks in which the currents or voltages are fixed in magnitude except for transient effects. In circuit 2 will turn our attention to the analysis of networks in which the magnitude of the source varies in a set manner. Of particular interest is the time-varying voltage that is commercially available in large quantities and is commonly called the *ac voltage.* (The letters *ac* are an abbreviation for *alternating current.*) To be absolutely rigorous, the terminology *ac voltage* or *ac current* is not sufficient to describe the type of signal we will be analyzing. Each waveform of Fig.1 is an **alternating waveform** available from commercial supplies.

The term *alternating* indicates only that the waveform alternates between two prescribed levels in a set time sequence (Fig.1).

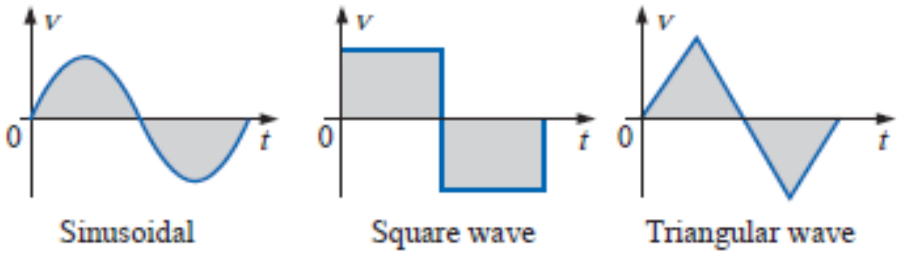


Fig.1 *Alternating waveforms.*

To be absolutely correct, the term *sinusoidal*, *square wave*, or *triangular* must also be applied. The pattern of particular interest is the **sinusoidal ac waveform** for voltage of Fig.1. Since this type of signal is encountered in the vast majority of instances, the abbreviated phrases *ac voltage* and *ac current* are commonly applied without confusion. For the other patterns of Fig.1, the descriptive term is always present, but frequently the *ac* abbreviation is dropped, resulting in the designation *square-wave* or *triangular* waveforms.

Definitions

The sinusoidal waveform of Fig.2 with its additional notation will now be used as a model in defining a few basic terms.

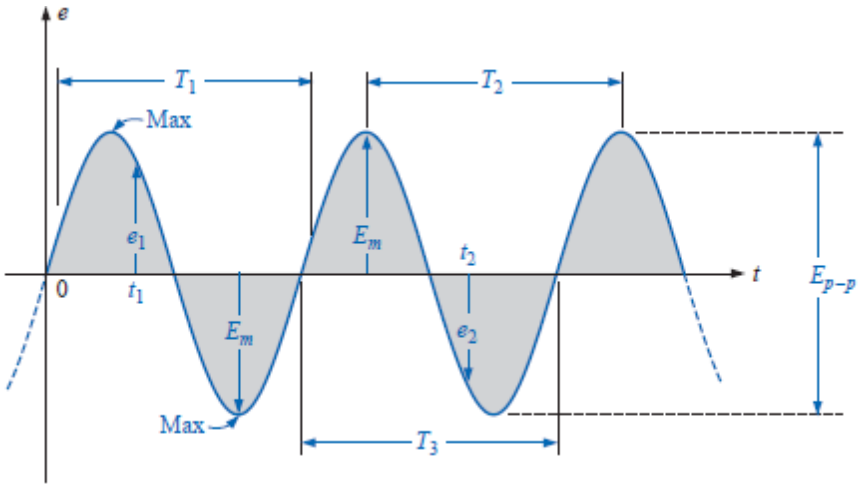


Fig.2 *Important parameters for a sinusoidal voltage.*

These terms, however, can be applied to any alternating waveform. It is important to remember as you proceed through the various definitions that the vertical scaling is in volts or amperes and the horizontal scaling is *always* in units of time.

Waveform: The variation of an ac voltage or current versus time is called its waveform. Since waveforms vary with time, they are designated by lowercase letters $v(t)$, $i(t)$, $e(t)$, and so on, rather than by uppercase letters V, I, and E as for dc. Often we drop the functional notation and simply use v, i, and e.

Instantaneous value: The magnitude of a waveform at any instant of time; denoted by lowercase letters (e_1, e_2).

Peak amplitude E_m : The maximum value of a waveform as measured from its *average*, or *mean*, value, denoted by uppercase letters (such as E_m for sources of voltage and V_m for the voltage drop across a load).

Peak value E_p: The maximum instantaneous value of a function as measured from the zero-volt level. For the waveform of Fig.2, the peak amplitude and peak value are the same, since the average value of the function is zero volts.

Peak-to-peak value: Denoted by $E_{p\text{-}p}$ or $V_{p\text{-}p}$, the full voltage between positive and negative peaks of the waveform, that is, the sum of the magnitude of the positive and negative peaks.

Periodic waveform: A waveform that continually repeats itself after the same time interval. The waveform of Fig.2 is a periodic waveform.

Period (*T*)**:** The time interval between successive repetitions of a periodic waveform (the period T_1 _ T_2 _ T_3 in Fig.2), as long as successive *similar points* of the periodic waveform are used in determining *T*.

Cycle: The portion of a waveform contained in *one period* of time. The cycles within T_1, T_2, and T_3 of Fig.2 may appear different in Fig.3, but they are all bounded by one period of time and therefore satisfy the definition of a cycle.

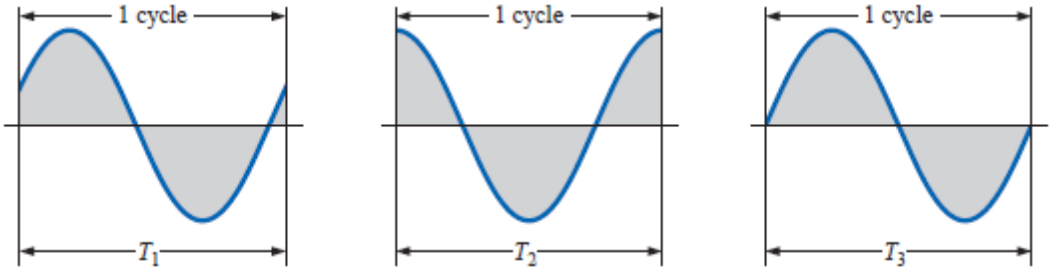


Fig.3 *Defining the cycle and period of a sinusoidal waveform.*

Frequency (*f*)**:** The number of cycles that occur in 1 s. The frequency of the waveform of Fig.4 (a) is 1 cycle per second, and for Fig.4 (b), 2 1⁄2 cycles per second. If a waveform of similar shape had a period of 0.5 s [Fig.4(c)], the frequency would be 2 cycles per second.

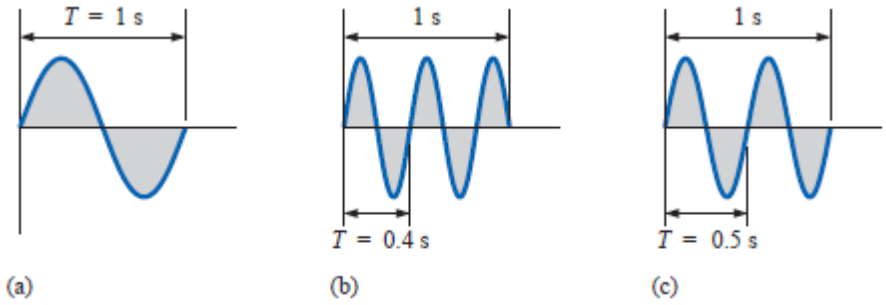


Fig.4 *Demonstrating the effect of a changing frequency on the period of a sinusoidal waveform.*

The unit of measure for frequency is the *hertz* (Hz), where:

1 hertz (HZ) = 1 cycle per second (c/s)

Since the frequency is inversely related to the period—that is, as one increase, the other decreases by an equal amount—the two can be related by the following equation:

$$f = \frac{1}{T}$$ f in Hz, T in second

and

$$T = \frac{1}{f}$$

Now define the angular frequency as: $\omega = 2\,\pi\,f$ rad/sec

And the angle $\theta = \omega t$ radians or degrees

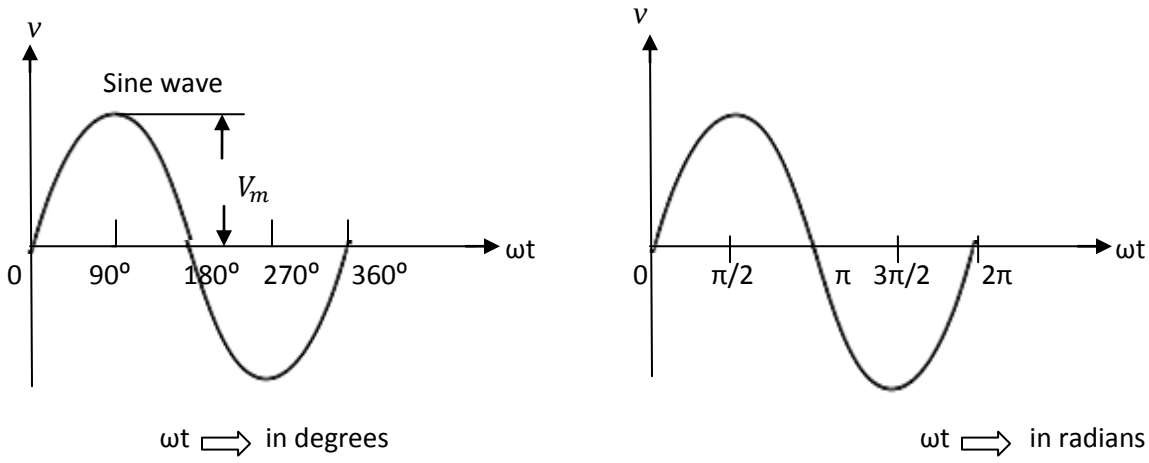
2. THE SINE WAVE

A sine wave is a sinusoidally varying quantity, or sinusoid, which can be expressed mathematically as

$$v = V_m \sin\,\omega t$$

or

$$v = V_m \sin\,\theta$$



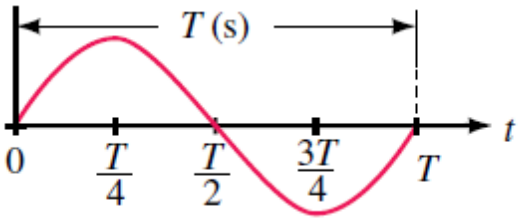


Fig.5 Sine wave.

Voltages and Currents as Functions of Time

Recall from Equation 1, $v = V_m \sin \omega t$.
Similarly

$$e = E_m \sin \omega t$$

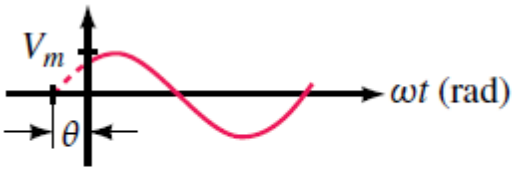
and

$$i = I_m \sin \omega t$$

Voltages and Currents with Phase Shifts

If a sine wave does not pass through zero at $t = 0$ s, it has a **phase shift.** Waveforms may be shifted to the left or to the right (see Fig.6). For a waveform shifted left as in (a),

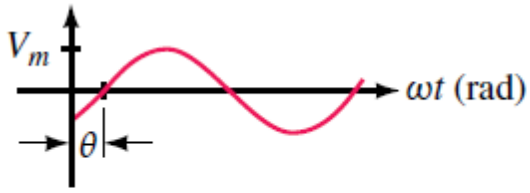
$$v = V_m \sin(\omega t + \theta)$$



(a) $v = V_m \sin(\omega t + \theta)$

while, for a waveform shifted right as in (b),

$$v = V_m \sin(\omega t - \theta)$$



(b) $v = V_m \sin(\omega t - \theta)$

Fig.6

Phase Difference

Phase difference refers to the angular displacement between different waveforms of the same frequency. Consider Figure 7. If the angular displacement is 0° as in (a), the waveforms are said to be **in phase;** otherwise, they are **out of phase.** When describing a phase difference, select one waveform as reference. Other waveforms then lead, lag, or are in phase with this reference.
For example, in (b), for reasons to be discussed in the next paragraph, the current waveform is said to lead the voltage waveform, while in (c) the current waveform is said to lag.

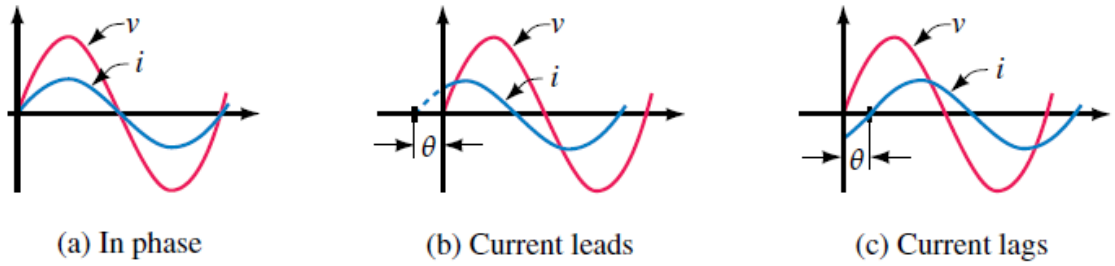


Fig. 7 Illustrating phase difference. In these examples, voltage is taken as reference.

Sometimes voltages and currents are expressed in terms of cos q*t* rather than sin q*t*. A cosine wave is a sine wave shifted by +90°, or alternatively, a sine wave is a cosine wave shifted by - 90°. For sines or cosines with an angle, the following formulas apply.

$$\cos(\omega t + \theta) = \sin(\omega t + \theta + 90°)$$

$$\sin(\omega t + \theta) = \cos(\omega t + \theta - 90°)$$

Average Value and RMS Value of Sine Wave

Because a sine wave is symmetrical, its area below the horizontal axis is the same as its area above the axis; thus, over a full cycle its net area is zero, independent of frequency and phase angle. Thus, the average of sin ω *t*, sin(ω*t* ± θ), sin 2ω*t*, cos ω *t*, cos (ω *t* ± θ), cos 2 ω *t*, and so on are each zero.

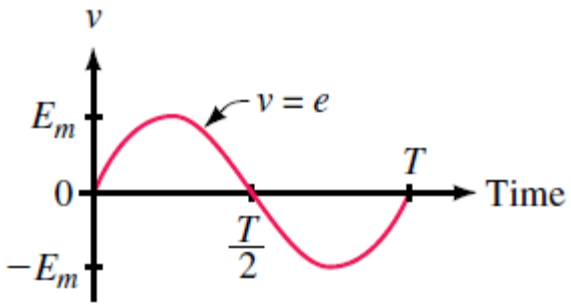
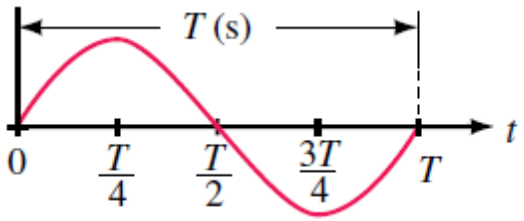


Fig.8 V_{avg} = 0 V

RMS Value

RMS values is also called **effective value .** *An effective value is an equivalent dc value: it tells you how many volts or amps of dc that a time-varying waveform is equal to in terms of its ability to produce average power.*



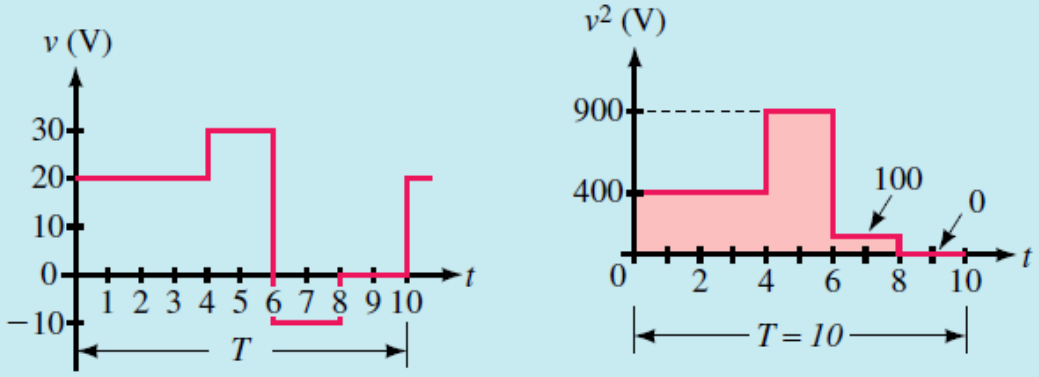
$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \quad (\sin\theta)^2 \quad dt}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

To compute effective values using this equation, do the following:
Step 1: Square the current (or voltage) curve.
Step 2: Find the area under the squared curve.

Step 3: Divide the area by the length of the curve.
Step 4: Find the square root of the value from Step 3.

EXAMPLE 15–28 One cycle of a voltage waveform is shown in Figure 15–65(a). Determine its effective value.



(a) Voltage waveform (b) Squared waveform

FIGURE 15–65

Solution Square the voltage waveform and plot it as in (b). Apply Equation 15–30:

$$V_{\text{eff}} = \sqrt{\frac{(400 \times 4) + (900 \times 2) + (100 \times 2) + (0 \times 2)}{10}}$$

$$= \sqrt{\frac{3600}{10}} = 19.0 \text{ V}$$

The waveform of Figure 15–65(a) has the same effective value as 19.0 V of steady dc.

EXAMPLE 15–29 Determine the effective value of the waveform of Figure 15–66(a).

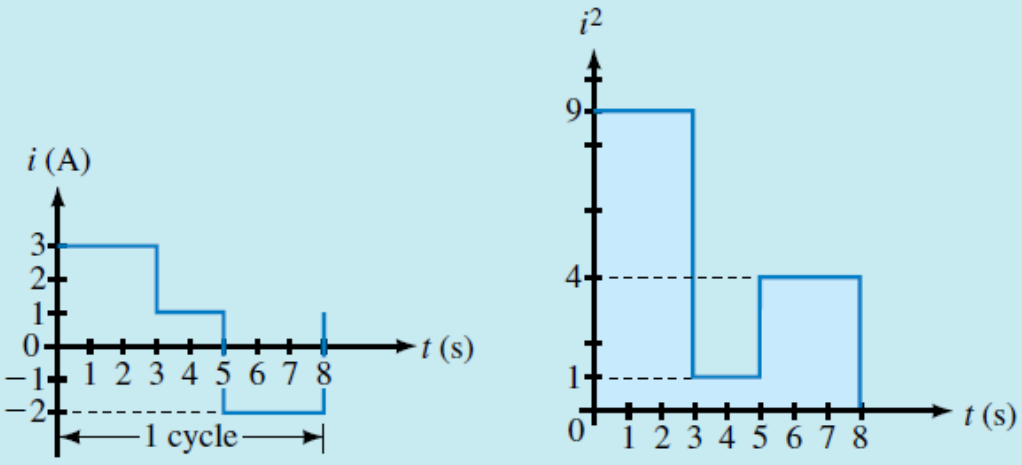


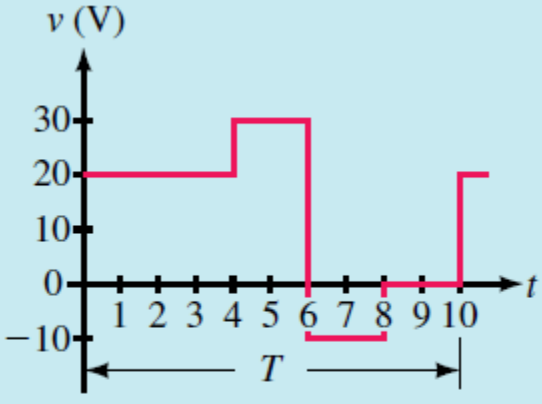
FIGURE 15–66

Solution Square the curve, then apply Equation 15–30. Thus,

$$I_{\text{eff}} = \sqrt{\frac{(9 \times 3) + (1 \times 2) + (4 \times 3)}{8}}$$

$$= \sqrt{\frac{41}{8}} = 2.26 \text{ A}$$

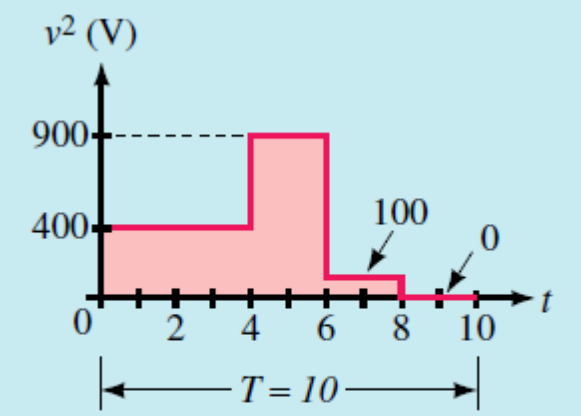
One cycle of a voltage waveform is shown in Figure 15(a). Determine its effective value.



(a) Voltage waveform

Solution Square the voltage waveform and plot it as in (b). Apply the following Equation:

$$v_{eff} = \sqrt{\frac{area\ under\ the\ curve\ of\ v^2}{base}}$$



(b) Squared waveform

$$V_{\text{eff}} = \sqrt{\frac{(400 \times 4) + (900 \times 2) + (100 \times 2) + (0 \times 2)}{10}}$$

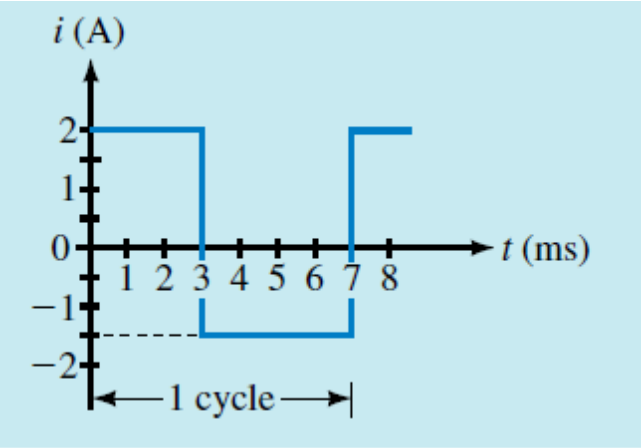
$$= \sqrt{\frac{3600}{10}} = 19.0\ \text{V}$$

a. Compute the average for the current waveform of Figure

b. If the negative portion of Figure 1 is - 3 A instead of -1.5 A, what is the average?
c. If the current is measured by a dc ammeter, what will the ammeter indicate?

d. Determine the effective value of the current . (Answer : 1.73 A)

Fig.1



Solution

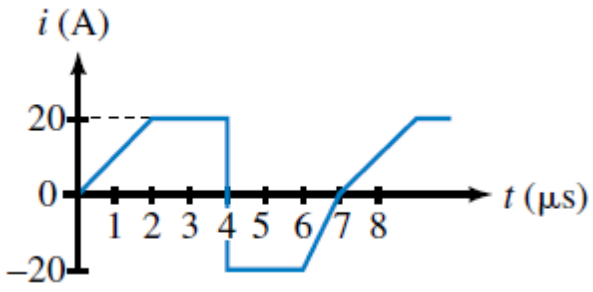
a. The waveform repeats itself after 7 ms. Thus, $T = 7$ ms and the average is

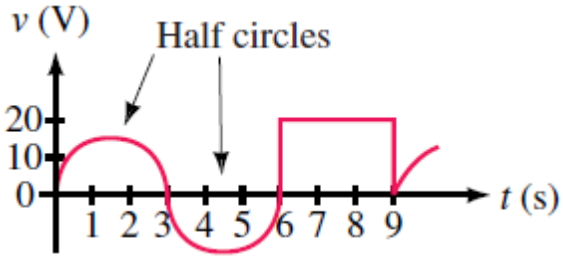
$$I_{avg} = \frac{(2\text{ A} \times 3\text{ ms}) - (1.5\text{ A} \times 4\text{ ms})}{7\text{ ms}} = \frac{6-6}{7} = 0\text{ A}$$

b. $$I_{avg} = \frac{(2\text{ A} \times 3\text{ ms}) - (3\text{ A} \times 4\text{ ms})}{7\text{ ms}} = \frac{-6\text{ A}}{7} = -0.857\text{ A}$$

c. A dc ammeter measuring (a) will indicate zero, while for (b) it will indicate −0.857 A.

Determine the averages for Figures 2(a) and (b).





Answers: a. 1.43 A b. 6.67 V

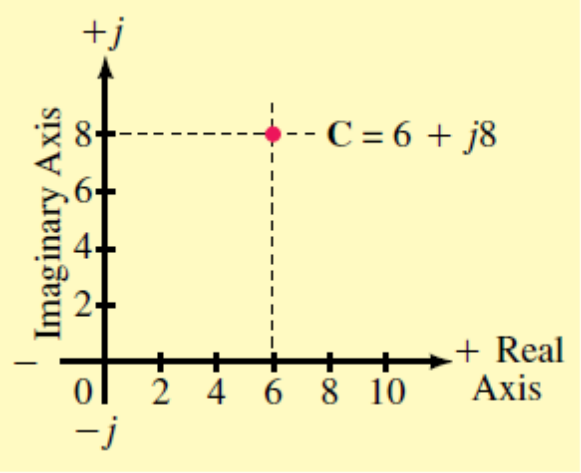
Complex Number Review

By 1893 , Steinmetz had reduced the very complex alternatingcurrent theory to a simple problem in algebra. The key concept in this simplification was the phasor—a representation based on complex numbers.
By representing voltages and currents as phasors, Steinmetz was able to define a quantity called **impedance** and then use it to determine voltage and current magnitude and phase relationships in one algebraic operation.

A **complex number** is a number of the form **C** = $a + jb$, where a and b are real numbers and $j = \sqrt{-1}$. The number a is called the **real** part of **C** and b is called its **imaginary** part. (In circuit theory, j is used to denote the imaginary component rather than i to avoid confusion with current i.)

Geometrical Representation
Complex numbers may be represented geometrically, either in rectangular form or in polar form as points on a two-dimensional plane called the **complex plane** (Figure 1). The complex number **C** = 6 + j8, for example, represents a point whose coordinate on the real axis is 6 and whose coordinate on the imaginary axis is 8. This form of representation is called the **rectangular form.**
Complex numbers may also be represented in **polar form** by magnitude and angle. Thus, **C** = $10\angle 53.13°$ (Figure 2) is a complex number with magnitude 10 and angle 53.13°. This magnitude and angle representation is just an alternate way of specifying the location of the point represented by **C** = $a + jb$.

FIGURE 1 A complex number in rectangular form.



Conversion between Rectangular and Polar Forms
To convert between forms, note from Figure 16–3 that
C= $a + jb$ (rectangular form) **(1)**
$\mathbf{C} = C\angle\theta$ (polar form) **(2)**
where C is the magnitude of **C.** From the geometry of the triangle,

$$a = C\cos\theta$$

$$b = C\sin\theta$$

where

$$C = \sqrt{a^2 + b^2}$$

and

$$\theta = \tan^{-1}\frac{b}{a}$$

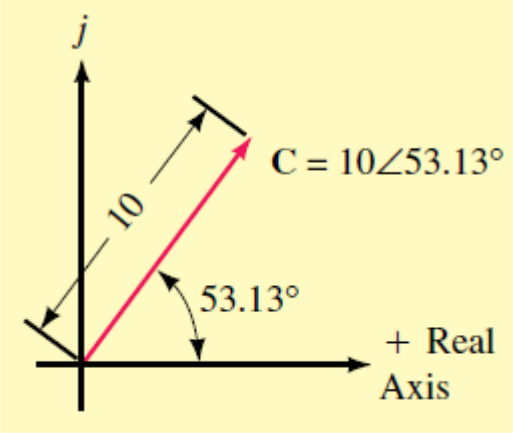


FIGURE 2 A complex number in polar form

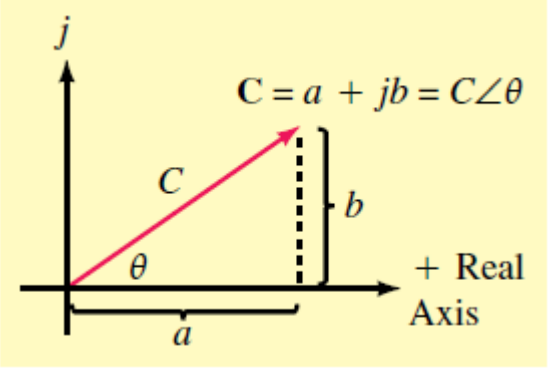
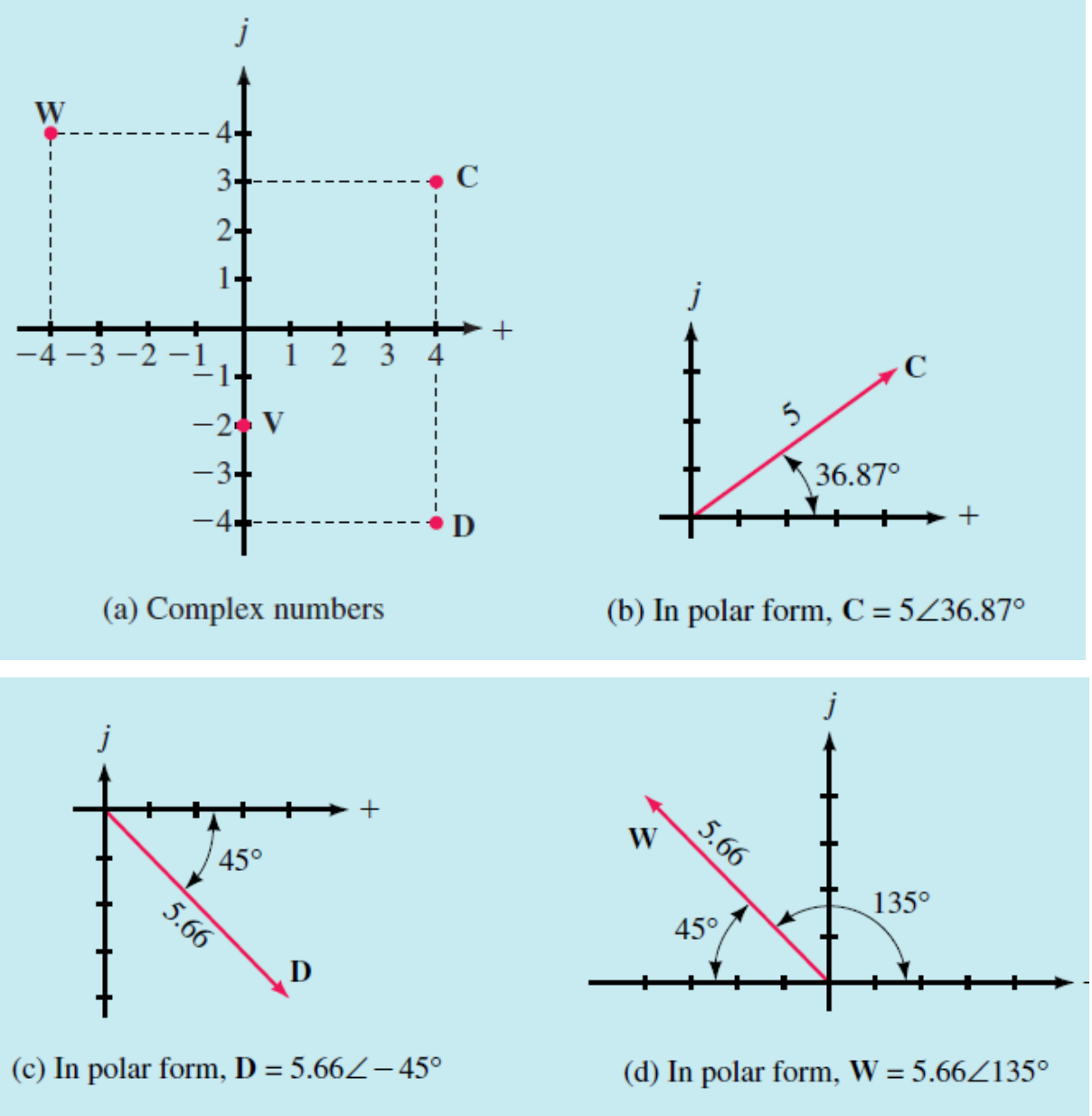


Fig.3 Polar and rectangular equivalence.

EXAMPLE 1 Determine rectangular and polar forms for the complex numbers **C, D, V,** and **W** of Figure 4(a).



(a) Complex numbers

(b) In polar form, $\mathbf{C} = 5\angle 36.87°$

(c) In polar form, $\mathbf{D} = 5.66\angle -45°$

(d) In polar form, $\mathbf{W} = 5.66\angle 135°$

Fig.4

Solution

Point C: Real part = 4; imaginary part = 3. Thus, $\mathbf{C} = 4 + j3$. In polar form, $C = \sqrt{4^2 + 3^2} = 5$ and $\theta_C = \tan^{-1}(3/4) = 36.87°$. Thus, $\mathbf{C} = 5\angle 36.87°$ as indicated in (b).

Point D: In rectangular form, $\mathbf{D} = 4 - j4$. Thus, $D = \sqrt{4^2 + 4^2} = 5.66$ and $\theta_D = \tan^{-1}(-4/4) = -45°$. Therefore, $\mathbf{D} = 5.66\angle -45°$, as shown in (c).

Point V: In rectangular form, $\mathbf{V} = -j2$. In polar form, $\mathbf{V} = 2\angle -90°$.

Point W: In rectangular form, $\mathbf{W} = -4 + j4$. Thus, $W = \sqrt{4^2 + 4^2} = 5.66$ and $\tan^{-1}(-4/4) = -45°$. Inspection of Figure 16–4(d) shows, however, that this 45° angle is the supplementary angle. The actual angle (measured from the positive horizontal axis) is 135°. Thus, $\mathbf{W} = 5.66\angle 135°$.

Powers of *j*

Powers of *j* are frequently required in calculations. Here are some useful powers:

$$j^2 = (\sqrt{-1})(\sqrt{-1}) = -1$$
$$j^3 = j^2 j = -j$$
$$j^4 = j^2 j^2 = (-1)(-1) = 1$$
$$(-j)j = 1$$
$$\frac{1}{j} = \frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = -j$$

Addition and Subtraction of Complex Numbers

Addition and subtraction of complex numbers can be performed analytically or graphically. Analytic addition and subtraction is most easily illustrated in rectangular form, while graphical addition and subtraction is best illustrated in polar form. For analytic addition, add real and imaginary parts separately.
Similarly for subtraction. For graphical addition, add vectorially as in Figure 5(a); for subtraction, change the sign of the subtrahend, then add, as in Figure 5(b).

EXAMPLE 2 Given **A** = 2 + *j*1 and **B** + 1 + *j*3. Determine their sum and difference analytically and graphically.

Solution

$$\mathbf{A} + \mathbf{B} = (2 + j1) + (1 + j3) = (2 + 1) + j(1 + 3) = 3 + j4.$$
$$\mathbf{A} - \mathbf{B} = (2 + j1) - (1 + j3) = (2 - 1) + j(1 - 3) = 1 - j2.$$

Graphical addition and subtraction are shown in Figure 16–5.

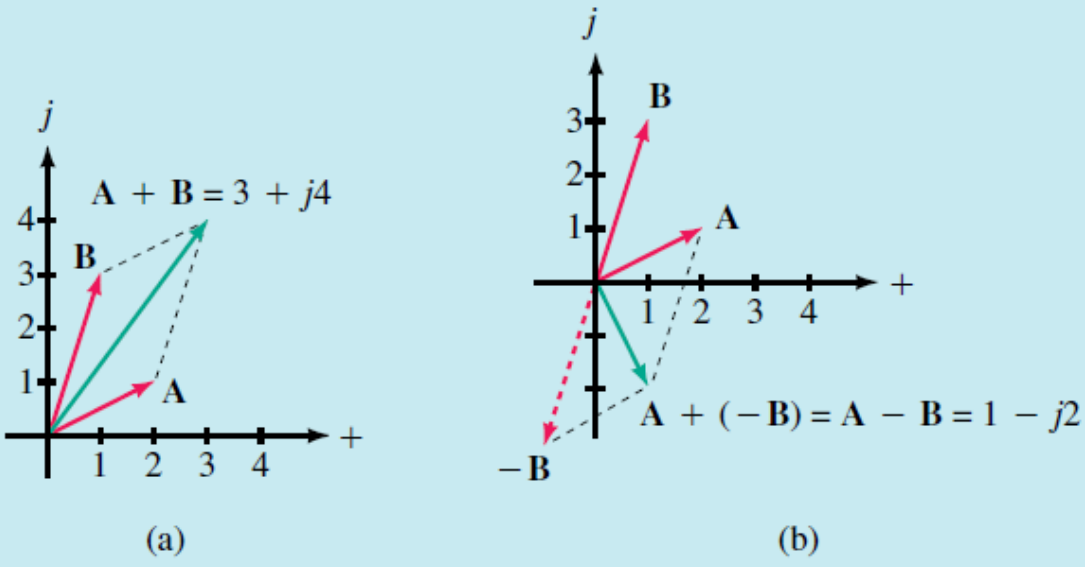


FIGURE 5

Multiplication and Division of Complex Numbers

These operations are usually performed in polar form. For multiplication, multiply magnitudes and add angles algebraically. For division, divide the magnitude of the denominator into the magnitude of the numerator, then subtract algebraically the angle of the denominator from that of the numerator.

Thus, given

$$\mathbf{A} = A\angle\theta_A \text{ and } \mathbf{B} = B\angle\theta_B,$$

$$\mathbf{A} \cdot \mathbf{B} = AB\underline{/\theta_A + \theta_B}$$

$$\mathbf{A}/\mathbf{B} = A/B\underline{/\theta_A - \theta_B}$$

EXAMPLE 16–3 Given $\mathbf{A} = 3\angle 35°$ and $\mathbf{B} = 2\angle -20°$, determine the product $\mathbf{A} \cdot \mathbf{B}$ and the quotient $\mathbf{A/B}$.

Solution

$$\mathbf{A} \cdot \mathbf{B} = (3\angle 35°)(2\angle -20°) = (3)(2)\underline{/35° - 20°} = 6\angle 15°$$

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(3\angle 35°)}{(2\angle -20°)} = \frac{3}{2}\underline{/35° - (-20°)} = 1.5\angle 55°$$

EXAMPLE 16–4 For computations involving purely real, purely imaginary, or small integer numbers, it is sometimes easier to multiply directly in rectangular form than it is to convert to polar. Compute the following directly:

a. $(-j3)(2 + j4)$.

b. $(2 + j3)(1 + j5)$.

Solution

a. $(-j3)(2 + j4) = (-j3)(2) + (-j3)(j4) = -j6 - j^2 12 = 12 - j6$

b. $(2 + j3)(1 + j5) = (2)(1) + (2)(j5) + (j3)(1) + (j3)(j5)$
$= 2 + j10 + j3 + j^2 15 = 2 + j13 - 15 = -13 + j13$

Series-Parallel ac Networks

1. VOLTAGE DIVIDER RULE

The basic format for the **voltage divider rule** in ac circuits is exactly the same as that for dc circuits:

$$\mathbf{V}_x = \frac{\mathbf{Z}_x\mathbf{E}}{\mathbf{Z}_T}$$

where $\mathbf{V}_x$ is the voltage across one or more elements in series that have total impedance $\mathbf{Z}_x$, $\mathbf{E}$ is the total voltage appearing across the series circuit, and $\mathbf{Z}_T$ is the total impedance of the series circuit.

EXAMPLE 1 Using the voltage divider rule, find the voltage across each element of the circuit of Fig. 1.

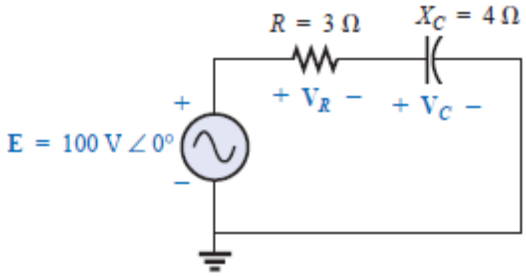


Fig.1

Solution:

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(4\ \Omega\angle -90°)(100\ \text{V}\angle 0°)}{4\ \Omega\angle -90° + 3\ \Omega\angle 0°} = \frac{400\angle -90°}{3 - j\,4}$$

$$= \frac{400\angle -90°}{5\angle -53.13°} = \mathbf{80\ V\angle -36.87°}$$

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(3\ \Omega\angle 0°)(100\ \text{V}\angle 0°)}{5\ \Omega\angle -53.13°} = \frac{300\angle 0°}{5\angle -53.13°}$$

$$= \mathbf{60\ V\angle +53.13°}$$

2. CURRENT DIVIDER RULE

The basic format for the **current divider rule** in ac circuits is exactly the same as that for dc circuits; that is, for two parallel branches with impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ as shown in Fig. 2,

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2\mathbf{I}_T}{\mathbf{Z}_1 + \mathbf{Z}_2} \quad \text{or} \quad \mathbf{I}_2 = \frac{\mathbf{Z}_1\mathbf{I}_T}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

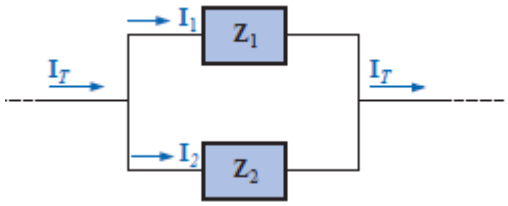
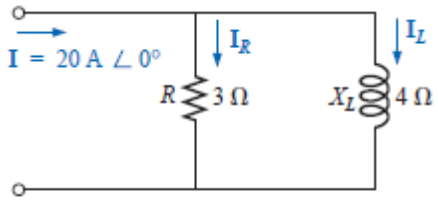


Fig.2

EXAMPLE 2 Using the current divider rule, find the current through each impedance of Fig. 3.

Fig.3



Solution:

$$\mathbf{I}_R = \frac{\mathbf{Z}_L\mathbf{I}_T}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(4\ \Omega\ \angle 90°)(20\ \text{A}\ \angle 0°)}{3\ \Omega\ \angle 0° + 4\ \Omega\ \angle 90°} = \frac{80\ \text{A}\angle 90°}{5\ \angle 53.13°}$$

$$= \mathbf{16\ A\ \angle 36.87°}$$

$$\mathbf{I}_L = \frac{\mathbf{Z}_R\mathbf{I}_T}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(3\ \Omega\ \angle 0°)(20\ \text{A}\ \angle 0°)}{5\ \Omega\ \angle 53.13°} = \frac{60\ \text{A}\angle 0°}{5\ \angle 53.13°}$$

$$= \mathbf{12\ A\ \angle -53.13°}$$

Series-Parallel ac Networks

EXAMPLE .1 For the network of Fig.1:

a. Calculate **Z***T*.
b. Determine **I***s*.
c. Calculate **V***R* and **V***C*.
d. Find **I***C*.
e. Compute the power delivered.
f. Find *Fp* of the network.

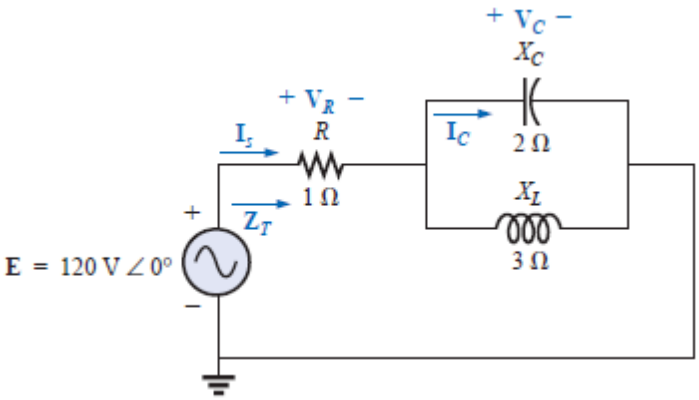


Fig.1

Solution

We re-draw the network as shown

In Fig.2

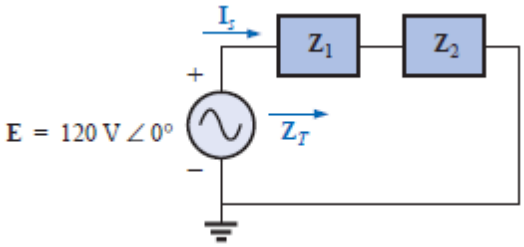


Fig.2

The total impedance is defined by

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2$$

with

$$\mathbf{Z}_1 = R\angle 0^\circ = 1\ \Omega\angle 0^\circ$$

$$\mathbf{Z}_2 = \mathbf{Z}_C \| \mathbf{Z}_L = \frac{(X_C\angle -90^\circ)(X_L\angle 90^\circ)}{-jX_C + jX_L} = \frac{(2\ \Omega\angle -90^\circ)(3\ \Omega\angle 90^\circ)}{-j\,2\ \Omega + j\,3\ \Omega}$$

$$= \frac{6\ \Omega\angle 0^\circ}{j\,1} = \frac{6\ \Omega\angle 0^\circ}{1\angle 90^\circ} = 6\ \Omega\angle -90^\circ$$

and

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 = 1\ \Omega - j\,6\ \Omega = \mathbf{6.08\ \Omega\angle -80.54^\circ}$$

b. $$\mathbf{I}_s = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{120\ \text{V}\angle 0^\circ}{6.08\ \Omega\angle -80.54^\circ} = \mathbf{19.74\ A\angle 80.54^\circ}$$

c. Referring to Fig.2, we find that $\mathbf{V}_R$ and $\mathbf{V}_C$ can be found by a direct application of Ohm's law:

$$\mathbf{V}_R = \mathbf{I}_s\mathbf{Z}_1 = (19.74\text{ A}\angle 80.54°)(1\ \Omega\angle 0°) = \mathbf{19.74\text{ V}\angle 80.54°}$$
$$\mathbf{V}_C = \mathbf{I}_s\mathbf{Z}_2 = (19.74\text{ A}\angle 80.54°)(6\ \Omega\angle -90°)$$
$$= \mathbf{118.44\text{ V}\angle -9.46°}$$

d. Now that $\mathbf{V}_C$ is known, the current $\mathbf{I}_C$ can also be found using Ohm's law.

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{\mathbf{Z}_C} = \frac{118.44\text{ V}\angle -9.46°}{2\ \Omega\angle -90°} = \mathbf{59.22\text{ A}\angle 80.54°}$$

e. $P_{del} = I_s^2R = (19.74\text{ A})^2(1\ \Omega) = \mathbf{389.67\text{ W}}$

f. $F_p = \cos\theta = \cos 80.54° = \mathbf{0.164\ leading}$

EXAMPLE .2 For the network of Fig.3:

a. If **I** is 50 A /_30°, calculate $\mathbf{I}_1$ using the current divider rule.
b. Repeat part (a) for $\mathbf{I}_2$.
c. Verify Kirchhoff's current law at one node.

Solutions:

a. Redrawing the circuit as in Fig.4, we have

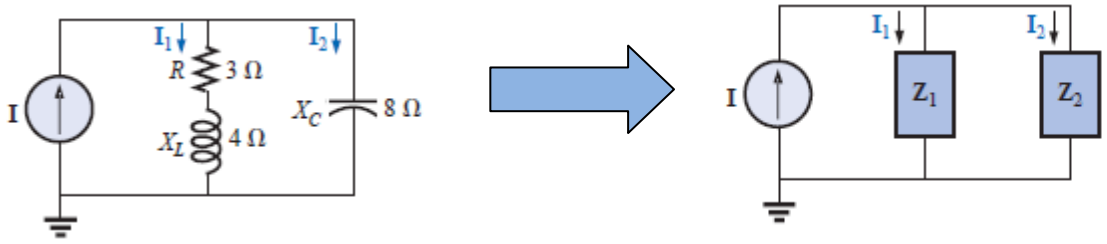


Fig.3 Fig.4

$$\mathbf{Z}_1 = R + jX_L = 3\ \Omega + j\,4\ \Omega = 5\ \Omega\angle 53.13°$$
$$\mathbf{Z}_2 = -jX_C = -j\,8\ \Omega = 8\ \Omega\angle -90°$$

Using the current divider rule yields

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2\mathbf{I}}{\mathbf{Z}_2 + \mathbf{Z}_1} = \frac{(8\ \Omega\angle -90°)(50\text{ A}\angle 30°)}{(-j\,8\ \Omega) + (3\ \Omega + j\,4\ \Omega)} = \frac{400\angle -60°}{3 - j\,4}$$
$$= \frac{400\angle -60°}{5\angle -53.13°} = \mathbf{80\text{ A}\angle -6.87°}$$

$$\text{b. } \mathbf{I}_2 = \frac{\mathbf{Z}_1\mathbf{I}}{\mathbf{Z}_2 + \mathbf{Z}_1} = \frac{(5\ \Omega\angle 53.13°)(50\ \text{A}\angle 30°)}{5\ \Omega\angle -53.13°} = \frac{250\angle 83.13°}{5\angle -53.13°}$$

$$= \mathbf{50\ A\angle 136.26°}$$

c. $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$

$$50\ \text{A}\angle 30° = 80\ \text{A}\angle -6.87° + 50\ \text{A}\angle 136.26°$$

$$= (79.43 - j\,9.57) + (-36.12 + j\,34.57)$$

$$= 43.31 + j\,25.0$$

$$50\ \text{A}\angle 30° = 50\ \text{A}\angle 30° \quad \text{(checks)}$$

EXAMPLE .3 For the network of Fig. 5:

a. Compute **I**.
b. Find $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
c. Verify Kirchhoff's current law by showing that

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3$$

d. Find the total impedance of the circuit.

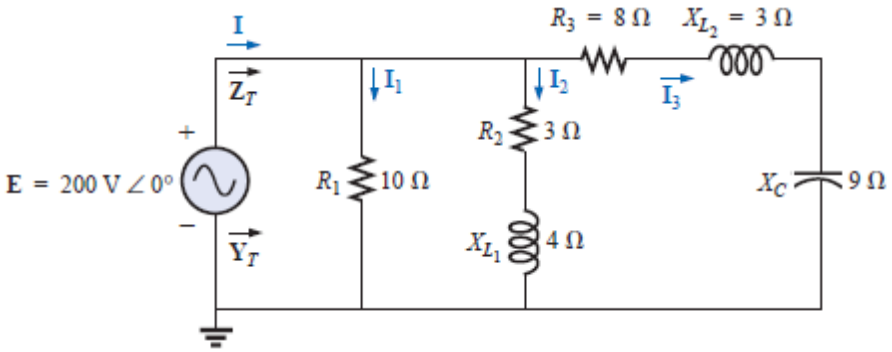


Fig.5

Solutions:

a. Redrawing the circuit as in Fig.6 reveals a strictly parallel network where

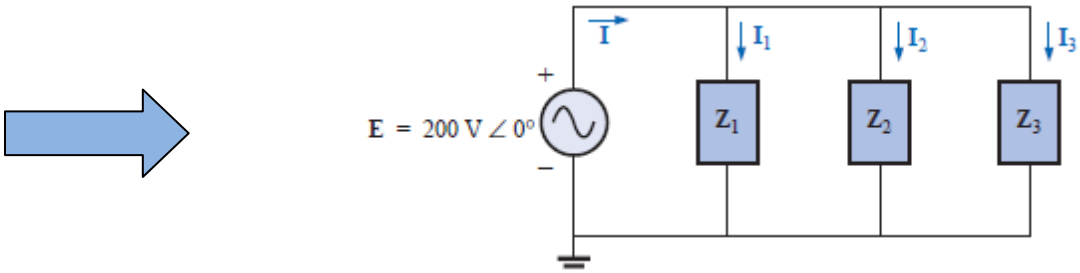


Fig.6

$$\mathbf{Z}_1 = R_1 = 10\ \Omega \angle 0°$$
$$\mathbf{Z}_2 = R_2 + jX_{L_1} = 3\ \Omega + j\,4\ \Omega$$
$$\mathbf{Z}_3 = R_3 + jX_{L_2} - jX_C = 8\ \Omega + j\,3\ \Omega - j\,9\ \Omega = 8\ \Omega - j\,6\ \Omega$$

The total admittance is

$$\mathbf{Y}_T = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3$$
$$= \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} = \frac{1}{10\ \Omega} + \frac{1}{3\ \Omega + j\,4\ \Omega} + \frac{1}{8\ \Omega - j\,6\ \Omega}$$
$$= 0.1\ \text{S} + \frac{1}{5\ \Omega \angle 53.13°} + \frac{1}{10\ \Omega \angle -36.87°}$$
$$= 0.1\ \text{S} + 0.2\ \text{S} \angle -53.13° + 0.1\ \text{S} \angle 36.87°$$
$$= 0.1\ \text{S} + 0.12\ \text{S} - j\,0.16\ \text{S} + 0.08\ \text{S} + j\,0.06\ \text{S}$$
$$= 0.3\ \text{S} - j\,0.1\ \text{S} = 0.316\ \text{S} \angle -18.435°$$

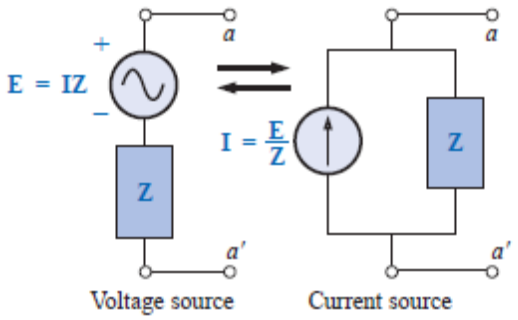
SOURCE CONVERSIONS

When applying the methods to be discussed, it may be necessary to convert a current source to a voltage source, or a voltage source to a current source. This **source conversion** can be accomplished in much the same manner as for dc circuits, except now we shall be dealing with phasors and impedances instead of just real numbers and resistors.

Independent Sources

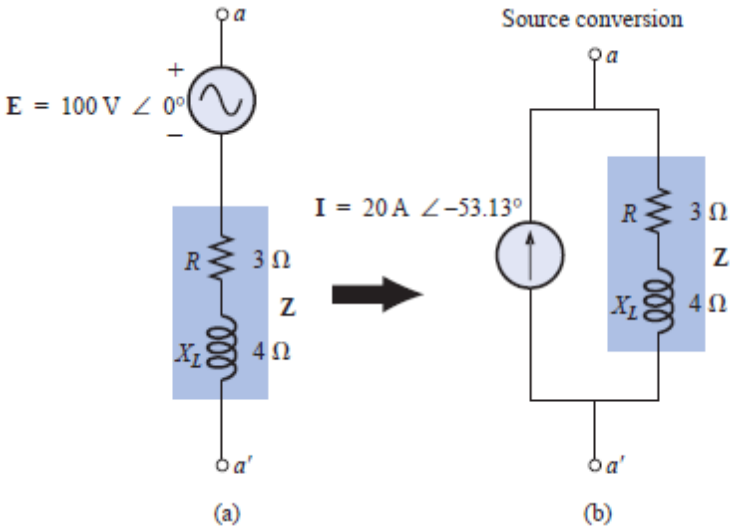
In general, the format for converting one type of independent source to another is as shown in Fig. 1.

Fig.1



EXAMPLE 1 Convert the voltage source of Fig. 2.(a) to a current

Solution:

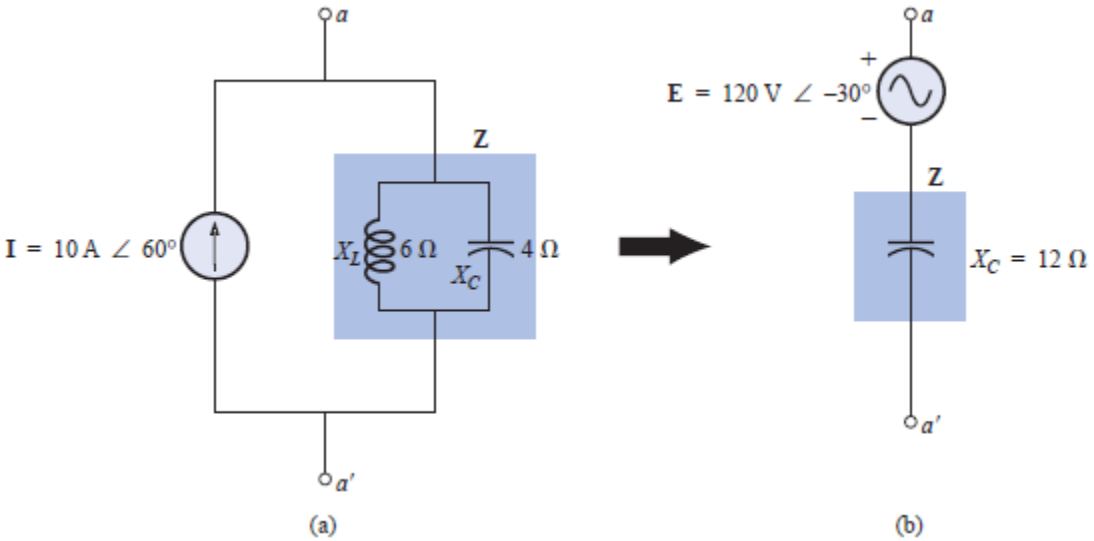


Solution:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{100\ \text{V} \angle 0°}{5\ \Omega \angle 53.13°}$$
$$= \mathbf{20\ A} \angle \mathbf{-53.13°}$$

Fig. 2(b)

EXAMPLE 2 Convert the current source of Fig. 3(a) to a voltage source.



Solution

$$\mathbf{Z} = \frac{\mathbf{Z}_C\mathbf{Z}_L}{\mathbf{Z}_C + \mathbf{Z}_L} = \frac{(X_C \angle -90°)(X_L \angle 90°)}{-j\,X_C + j\,X_L}$$

$$= \frac{(4\ \Omega \angle -90°)(6\ \Omega \angle 90°)}{-j\,4\ \Omega + j\,6\ \Omega} = \frac{24\ \Omega \angle 0°}{2 \angle 90°}$$

$$= \mathbf{12\ \Omega \angle -90°} \quad \text{[Fig. 3 (b)]}$$

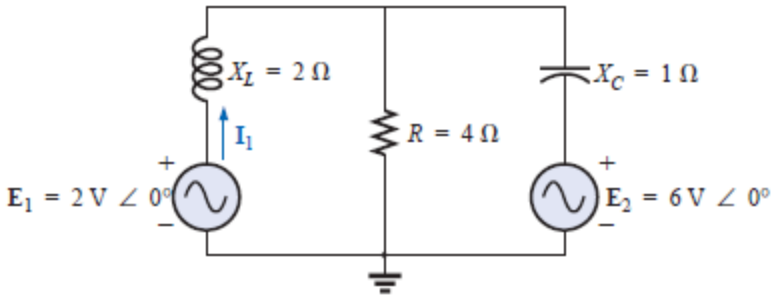
$$\mathbf{E} = \mathbf{IZ} = (10\ \text{A} \angle 60°)(12\ \Omega \angle -90°)$$

$$= \mathbf{120\ V \angle -30°} \quad \text{[Fig. 3 (b)]}$$

MESH ANALYSIS

EXAMPLE 1 Using the general approach to mesh analysis, find the current $\mathbf{I}_1$ in Fig. 1.

Fig.1



Solution: When applying these methods to ac circuits, it is good practice to represent the resistors and reactances (or combinations thereof) by subscripted impedances. When the total solution is found in terms of these subscripted impedances, the numerical values can be substituted to find the unknown quantities.
The network is redrawn in Fig. 2 with subscripted impedances:

$\mathbf{Z}_1 = +j\,X_L = +j\,2\ \Omega$ $\quad \mathbf{E}_1 = 2\ \text{V} \angle 0°$

$\mathbf{Z}_2 = R = 4\ \Omega$ $\quad \mathbf{E}_2 = 6\ \text{V} \angle 0°$

$\mathbf{Z}_3 = -j\,X_C = -j\,1\ \Omega$

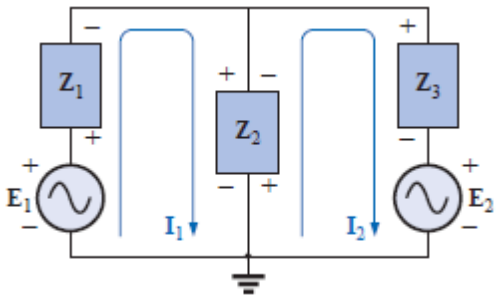


Fig.2 *Assigning the mesh currents and Subscripted impedances for the network of Fig. 1.*

Steps of solution:

1. Assign a current in the clockwise direction to each independent closed loop of the network. It is not necessary to choose the clockwise direction for each loop current.

2. Indicate the polarities within each loop for each impedance as determined by the assumed direction of loop current for that loop.
Steps 1 and 2 are as indicated in Fig. 2.

3. Apply Kirchhoff's voltage law around each closed loop in the clockwise direction.

$$+\mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0$$
$$-\mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_1) - \mathbf{I}_2\mathbf{Z}_3 - \mathbf{E}_2 = 0$$

or

$$\mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{I}_1\mathbf{Z}_2 + \mathbf{I}_2\mathbf{Z}_2 = 0$$
$$-\mathbf{I}_2\mathbf{Z}_2 + \mathbf{I}_1\mathbf{Z}_2 - \mathbf{I}_2\mathbf{Z}_3 - \mathbf{E}_2 = 0$$

so that

$$\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 = \mathbf{E}_1$$
$$\mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{I}_1\mathbf{Z}_2 = -\mathbf{E}_2$$

which are rewritten as

$$\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 = \mathbf{E}_1$$
$$-\mathbf{I}_1\mathbf{Z}_2 + \mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) = -\mathbf{E}_2$$

Using determinants, we obtain

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{E}_1 & -\mathbf{Z}_2 \\ -\mathbf{E}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_2 & -\mathbf{Z}_2 \\ -\mathbf{Z}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}}$$
$$= \frac{\mathbf{E}_1(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{E}_2(\mathbf{Z}_2)}{(\mathbf{Z}_1 + \mathbf{Z}_2)(\mathbf{Z}_2 + \mathbf{Z}_3) - (\mathbf{Z}_2)^2}$$
$$= \frac{(\mathbf{E}_1 - \mathbf{E}_2)\mathbf{Z}_2 + \mathbf{E}_1\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}$$

Substituting numerical values yields

$$\mathbf{I}_1 = \frac{(2\,\text{V} - 6\,\text{V})(4\,\Omega) + (2\,\text{V})(-j\,1\,\Omega)}{(+j\,2\,\Omega)(4\,\Omega) + (+j\,2\,\Omega)(-j\,2\,\Omega) + (4\,\Omega)(-j\,2\,\Omega)}$$
$$= \frac{-16 - j\,2}{j\,8 - j^2 2 - j\,4} = \frac{-16 - j\,2}{2 + j\,4} = \frac{16.12\,\text{A}\angle -172.87°}{4.47\angle 63.43°}$$
$$= \mathbf{3.61\,A\angle -236.30°} \quad \text{or} \quad \mathbf{3.61\,A\angle 123.70°}$$

Appendix A : DETERMINANTS

Consider the following equations, where x and y are the unknown variables and $a1$, $a2$, $b1$, $b2$, $c1$, and $c2$ are constants:

Col. 1		Col. 2		Col. 3	
a_1x	+	b_1y	=	c_1	(1)
a_2x	+	b_2y	=	c_2	(2)

It is certainly possible to solve for one variable in Eq. (1) and substitute into Eq. (2). That is, solving for x in Eq. (1),

$$x = \frac{c_1 - b_1y}{a_1}$$

and substituting the result in Eq. (2),

$$a_2\left(\frac{c_1 - b_1y}{a_1}\right) + b_2y = c_2$$

- Using determinants to solve for x and y requires that the following formats be established for each variable:

Col. 1	Col. 2		Col. 1	Col. 2
$x = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$		$y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$		(3)

The determinant is :

Col. 1	Col. 2		
Determinant $= D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$			(4)

Solution for x and y is :

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad (5)$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \qquad (6)$$

Example A1 Solve for x and y :

$$\begin{aligned} 2x + \ y &= 3 \\ 3x + 4y &= 2 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}} = \frac{(3)(4) - (2)(1)}{(2)(4) - (3)(1)} = \frac{12 - 2}{8 - 3} = \frac{10}{5} = \mathbf{2}$$

$$y = \frac{\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}{5} = \frac{(2)(2) - (3)(3)}{5} = \frac{4 - 9}{5} = \frac{-5}{5} = \mathbf{-1}$$

- Consider the three following simultaneous equations:

Col. 1		Col. 2		Col. 3		Col. 4
a_1x	+	b_1y	+	c_1z	=	d_1
a_2x	+	b_2y	+	c_2z	=	d_2
a_3x	+	b_3y	+	c_3z	=	d_3

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$$

where $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

(4(−) 5(−) 6(−); 1(+) 2(+) 3(+))

D = $+(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$

Warning: **This method of expansion is good only for third-order determinants!** It cannot be applied to fourth- and higher-order systems.

EXAMPLE A2 Evaluate the following determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 0 & 4 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 0 & 4 & 2 \end{vmatrix} \begin{matrix} 1 & 2 \\ -2 & 1 \\ 0 & 4 \end{matrix}$$

(−) (−) (−); (+) (+) (+)

Solution:

$$[(1)(1)(2) + (2)(0)(0) + (3)(-2)(4)] - [(0)(1)(3) + (4)(0)(1) + (2)(-2)(2)]$$
$$= (2 + 0 - 24) - (0 + 0 - 8) = (-22) - (-8)$$
$$= -22 + 8 = \mathbf{-14}$$

EXAMPLE A3 Solve for *x*, *y*, and *z*:

$$1x + 0y - 2z = -1$$
$$0x + 3y + 1z = +2$$
$$1x + 2y + 3z = 0$$

Solution

$$x = \frac{\begin{vmatrix} -1 & 0 & -2 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{vmatrix} \begin{matrix} -1 & 0 \\ 2 & 3 \\ 0 & 2 \end{matrix}}{\begin{vmatrix} 1 & 0 & -2 \\ 0 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} 1 & 0 \\ 0 & 3 \\ 1 & 2 \end{matrix}}$$

$$= \frac{[(-1)(3)(3) + (0)(1)(0) + (-2)(2)(2)] - [(0)(3)(-2) + (2)(1)(-1) + (3)(2)(0)]}{[(1)(3)(3) + (0)(1)(1) + (-2)(0)(2)] - [(1)(3)(-2) + (2)(1)(1) + (3)(0)(0)]}$$

$$= \frac{(-9 + 0 - 8) - (0 - 2 + 0)}{(9 + 0 + 0) - (-6 + 2 + 0)}$$

$$= \frac{-17 + 2}{9 + 4} = -\frac{\mathbf{15}}{\mathbf{13}}$$

$$y = \frac{\begin{vmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix}}{13}$$

$$= \frac{[(1)(2)(3) + (-1)(1)(1) + (-2)(0)(0)] - [(1)(2)(-2) + (0)(1)(1) + (3)(0)(-1)]}{13}$$

$$= \frac{5 + 4}{13} = \frac{\mathbf{9}}{\mathbf{13}}$$

To solve for z: from $0x + 3y + 1z = +2$,

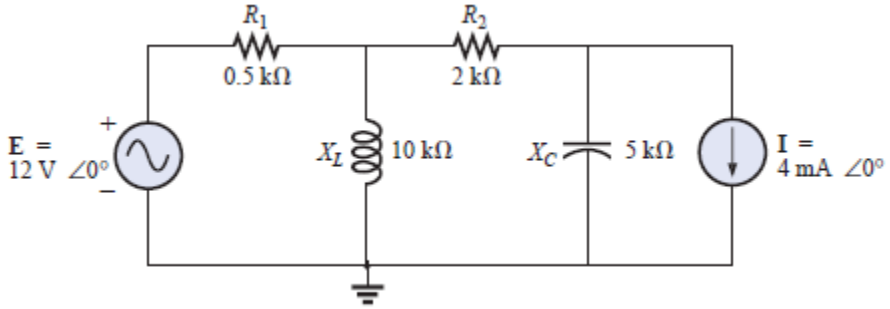
$$z = 2 - 3y = 2 - 3\left(\frac{9}{13}\right) = \frac{26}{13} - \frac{27}{13} = -\frac{\mathbf{1}}{\mathbf{13}}$$

NODAL ANALYSIS

1. Determine the number of nodes within the network.
2. Pick a reference node and label each remaining node with a subscripted value of voltage: $\mathbf{V_1}$, $\mathbf{V_2}$***, and so on.***
3. Apply Kirchhoff's current law at each node except the reference. Assume that all unknown currents leave the node for each application of Kirchhoff's current law.
4. Solve the resulting equations for the nodal voltages.

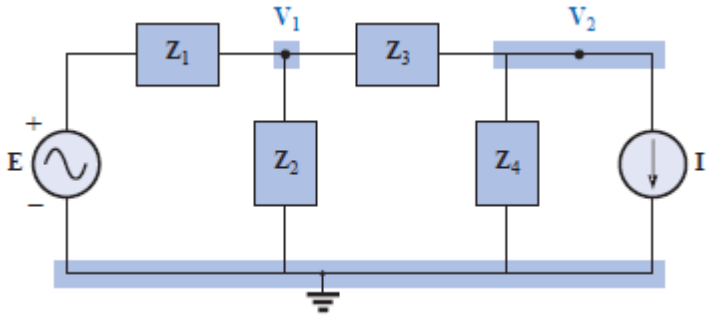
EXAMPLE 2 Determine the voltage across the inductor for the network of Fig. 3.

Fig.3



Solution 1:
Steps 1 and 2 are as indicated in Fig. 4.

Fig.4



Step 3: Note Fig. 5 for the application of Kirchhoff's current law to node $\mathbf{V}_1$:

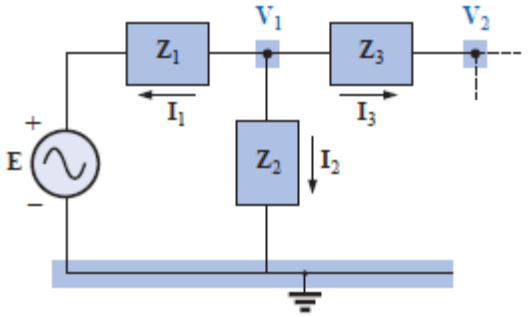


Fig.5

$$\Sigma\, \mathbf{I}_i = \Sigma\, \mathbf{I}_o$$

$$0 = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3$$

$$\frac{\mathbf{V}_1 - \mathbf{E}}{\mathbf{Z}_1} + \frac{\mathbf{V}_1}{\mathbf{Z}_2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{\mathbf{Z}_3} = 0$$

Rearranging terms:

$$\mathbf{V}_1\left[\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}\right] - \mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3}\right] = \frac{\mathbf{E}_1}{\mathbf{Z}_1}$$

Note Fig. 6 for the application of Kirchhoff's current law to node **V**2.

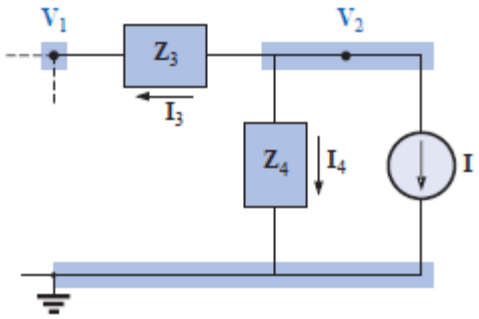


Fig.6

$$0 = \mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}$$

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{\mathbf{Z}_3} + \frac{\mathbf{V}_2}{\mathbf{Z}_4} + \mathbf{I} = 0$$

Applying Kirchhoff's current law to the node **V**2 *of Fig. 4.*

$$\mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4}\right] - \mathbf{V}_1\left[\frac{1}{\mathbf{Z}_3}\right] = -\mathbf{I}$$

$$\mathbf{V}_1\left[\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}\right] - \mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3}\right] = \frac{\mathbf{E}}{\mathbf{Z}_1}$$

$$\mathbf{V}_1\left[\frac{1}{\mathbf{Z}_3}\right] - \mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4}\right] = \mathbf{I}$$

$$\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} = \frac{1}{0.5\ \text{k}\Omega} + \frac{1}{j\ 10\ \text{k}\Omega} + \frac{1}{2\ \text{k}\Omega} = 2.5\ \text{mS}\ \angle -2.29°$$

$$\frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4} = \frac{1}{2\ \text{k}\Omega} + \frac{1}{-j\ 5\ \text{k}\Omega} = 0.539\ \text{mS}\ \angle 21.80°$$

and

$$\begin{aligned} \mathbf{V}_1[2.5\ \text{mS}\ \angle -2.29^\circ] - \mathbf{V}_2[0.5\ \text{mS}\ \angle 0^\circ] &= 24\ \text{mA}\ \angle 0^\circ \\ \mathbf{V}_1[0.5\ \text{mS}\ \angle 0^\circ] - \mathbf{V}_2[0.539\ \text{mS}\ \angle 21.80^\circ] &= 4\ \text{mA}\ \angle 0^\circ \end{aligned}$$

with

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 24\ \text{mA}\ \angle 0^\circ & -0.5\ \text{mS}\ \angle 0^\circ \\ 4\ \text{mA}\ \angle 0^\circ & -0.539\ \text{mS}\ \angle 21.80^\circ \end{vmatrix}}{\begin{vmatrix} 2.5\ \text{mS}\ \angle -2.29^\circ & -0.5\ \text{mS}\ \angle 0^\circ \\ 0.5\ \text{mS}\ \angle 0^\circ & -0.539\ \text{mS}\ \angle 21.80^\circ \end{vmatrix}}$$

$$= \frac{(24\ \text{mA}\ \angle 0^\circ)(-0.539\ \text{mS}\ \angle 21.80^\circ) + (0.5\ \text{mS}\ \angle 0^\circ)(4\ \text{mA}\ \angle 0^\circ)}{(2.5\ \text{mS}\ \angle -2.29^\circ)(-0.539\ \text{mS}\ \angle 21.80^\circ) + (0.5\ \text{mS}\ \angle 0^\circ)(0.5\ \text{mS}\ \angle 0^\circ)}$$

$$= \frac{-12.94 \times 10^{-6}\ \text{V}\ \angle 21.80^\circ + 2 \times 10^{-6}\ \text{V}\ \angle 0^\circ}{-1.348 \times 10^{-6}\ \angle 19.51^\circ + 0.25 \times 10^{-6}\ \angle 0^\circ}$$

$$= \frac{-(12.01 + j\,4.81) \times 10^{-6}\ \text{V} + 2 \times 10^{-6}\ \text{V}}{-(1.271 + j\,0.45) \times 10^{-6} + 0.25 \times 10^{-6}}$$

$$= \frac{-10.01\ \text{V} - j\,4.81\ \text{V}}{-1.021 - j\,0.45} = \frac{11.106\ \text{V}\ \angle -154.33^\circ}{1.116\ \angle -156.21^\circ}$$

$$\mathbf{V}_1 = \mathbf{9.95\ V\ \angle 1.88^\circ}$$

AC Power Calculations

AVERAGE POWER AND POWER FACTOR

For any load in a sinusoidal ac network, the voltage across the load and the current through the load will vary in a sinusoidal nature. The questions then arise, How does the power to the load determined by the product $v \cdot i$ vary, and what fixed value can be assigned to the power since it will vary with time?
If we take the general case depicted in Fig. 1 and use the following for v and i:

$$v = V_m \sin(\omega t + \theta_v)$$
$$i = I_m \sin(\omega t + \theta_i)$$

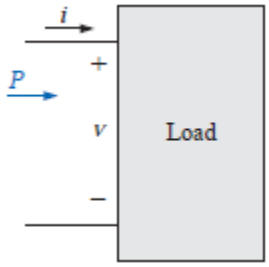


Fig.1

then the power is defined by

$$p = vi = V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i)$$
$$= V_m I_m \sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$$

Using the trigonometric identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

the function $\sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$ becomes

$$\sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$$
$$= \frac{\cos[(\omega t + \theta_v) - (\omega t + \theta_i)] - \cos[(\omega t + \theta_v) + (\omega t + \theta_i)]}{2}$$
$$= \frac{\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i)}{2}$$

so that

$$p = \overbrace{\left[\frac{V_m I_m}{2} \cos(\theta_v - \theta_i)\right]}^{\text{Fixed value}} - \overbrace{\left[\frac{V_m I_m}{2} \cos(2\omega t + \theta_v + \theta_i)\right]}^{\text{Time-varying (function of } t\text{)}}$$

A plot of v, i, and p on the same set of axes is shown in Fig. 2. Note that the second factor in the preceding equation is a cosine wave with an amplitude of $V_m I_m/2$ and with a frequency twice that of the voltage or current. The average value of this term is zero over one cycle, producing no net transfer of energy in any one direction.

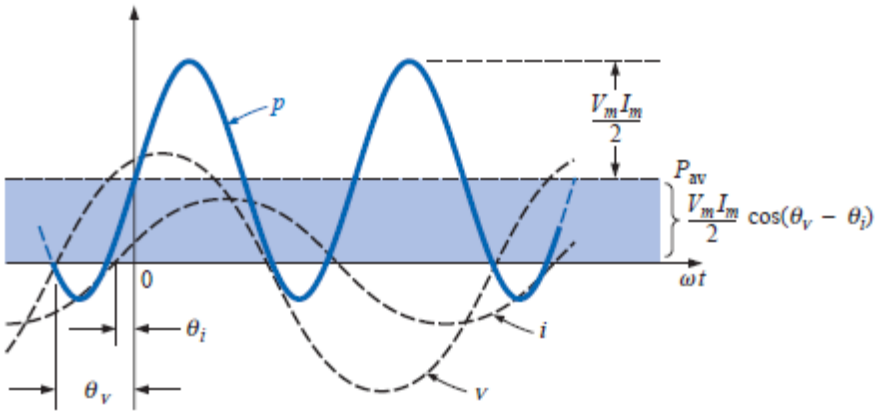


Fig.2

The first term in the preceding equation, however, has a constant magnitude (no time dependence) and therefore provides some net transfer of energy. This term is referred to as the **average power,** the reason for which is obvious from Fig. 2. The average power, or **real power** as it is sometimes called, is the power delivered to and dissipated by the load. It corresponds to the power calculations performed for dc networks. The angle ($\theta_v - \theta_i$) is the phase angle between *v* and *i*. Since cos(-a) = cos a,

the magnitude of average power delivered is independent of whether v leads i or i leads v.

Defining as equal to $|\theta_v - \theta_i|$, where | |indicates that only the magnitude is important and the sign is immaterial, we have

$$P = \frac{V_m I_m}{2}\cos\theta \qquad \text{(watts, W)}$$

where *P* is the average power in watts. This equation can also be written

$$P = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

or, since $V_{eff} = \frac{V_m}{\sqrt{2}}$ and $I_{eff} = \frac{I_m}{\sqrt{2}}$

Hence $$P = V_{eff} I_{eff}\cos\theta$$

- ## Resistor

In a purely resistive circuit, since *v* and *i* are in phase, $|\theta_v - \theta_i| = \theta = 0°$, and $\cos\theta = \cos 0° = 1$, so that

$$P = \frac{V_m I_m}{2} = V_{eff} I_{eff} \qquad \text{(W)}$$

Or, since $I_{eff} = \frac{V_{eff}}{R}$

then $$P = \frac{V_{eff}^2}{R} = I_{eff}^2 R \qquad \text{(W)}$$

- **Inductor**

In a purely inductive circuit, since *v* leads *i* by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2}\cos 90° = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal inductor (no associated resistance) is zero watts.

- **Capacitor**

In a purely capacitive circuit, since *i* leads *v* by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2}\cos(90°) = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal capacitor (no associated resistance) is zero watts.

EXAMPLE 1 Find the average power dissipated in a network whose input current and voltage are the following:

$i = 5\sin(\omega t + 40°)$

$v = 10\sin(\omega t + 40°)$

Solution: Since *v* and *i* are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

$$P = \frac{V_m I_m}{2} = \frac{(10\text{ V})(5\text{ A})}{2} = \mathbf{25\ W}$$

or

$$R = \frac{V_m}{I_m} = \frac{10\text{ V}}{5\text{ A}} = 2\ \Omega$$

and

$$P = \frac{V_{\text{eff}}^2}{R} = \frac{[(0.707)(10\text{ V})]^2}{2} = \mathbf{25\ W}$$

or

$$P = I_{\text{eff}}^2 R = [(0.707)(5\text{ A})]^2(2) = \mathbf{25\ W}$$

EXAMPLE 2: Determine the average power delivered to networks having the following input voltage and current:

a. $v = 100\sin(\omega t + 40°)$
   $i = 20\sin(\omega t + 70°)$
b. $v = 150\sin(\omega t - 70°)$
   $i = 3\sin(\omega t - 50°)$

Solutions:

a. $V_m = 100, \quad \theta_v = 40°$
$I_m = 20, \quad \theta_i = 70°$
$\theta = |\theta_v - \theta_i| = |40° - 70°| = |-30°| = 30°$
and

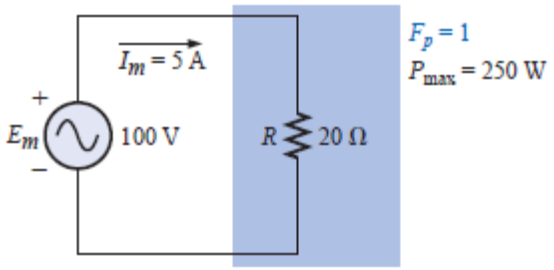
$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(100\text{ V})(20\text{ A})}{2}\cos(30°) = (1000\text{ W})(0.866)$$
$$= \mathbf{866\text{ W}}$$

b. $V_m = 150\text{ V}, \quad \theta_v = -70°$
$I_m = 3\text{ A}, \quad \theta_i = -50°$
$\theta = |\theta_v - \theta_i| = |-70° - (-50°)|$
$= |-70° + 50°| = |-20°| = 20°$
and

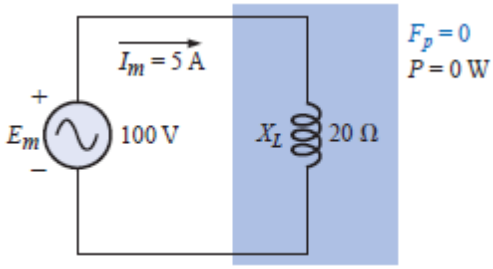
$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(150\text{ V})(3\text{ A})}{2}\cos(20°) = (225\text{ W})(0.9397)$$
$$= \mathbf{211.43\text{ W}}$$

ed power level is
cos θ= 1, the power delivered is a maximum. Since it has such control, the expression was given the name **power factor** and is defined by

$$\text{Power factor} = F_p = \cos\theta$$



Purely resistive load with $F_p = 1$.



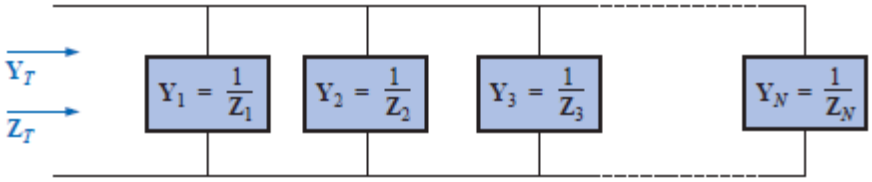
Purely inductive load with $F_p = 0$.

ADMITTANCE AND SUSCEPTANCE

The discussion for **parallel ac circuits** will be very similar to that for dc circuits. In dc circuits, *conductance* (*G*) was defined as being equal to 1/*R*. The total conductance of a parallel circuit was then found by adding the conductance of each branch. The total resistance R_T is simply $1/G_T$.
In ac circuits, we define **admittance** (**Y**) as being equal to 1/**Z**. The unit of measure for admittance as defined by the SI system is *siemens*, which has the symbol S. Admittance is a measure of how well an ac circuit will *admit*, or allow, current to flow in the circuit. The larger its value, therefore, the heavier the

current flow for the same applied potential. The total admittance of a circuit can also be found by finding the sum of the parallel admittances. The total impedance $\mathbf{Z}_T$ of the circuit is then $1/\mathbf{Y}_T$; that is, for the network of Fig. 15.54:

$$\mathbf{Y}_T = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \cdots + \mathbf{Y}_N$$



or, since $\mathbf{Z} = 1/\mathbf{Y}$,

$$\frac{1}{\mathbf{Z}_T} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots + \frac{1}{\mathbf{Z}_N}$$

For two impedances in parallel,

or

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

As pointed out in the introduction to this section, conductance is the reciprocal of resistance, and

$$\mathbf{Y}_R = \frac{1}{\mathbf{Z}_R} = \frac{1}{R\angle 0^\circ} = G\angle 0^\circ$$

The reciprocal of reactance ($1/X$) is called **susceptance** and is a measure of how *susceptible* an element is to the passage of current through it. Susceptance is also measured in *siemens* and is represented by the capital letter *B*.
For the inductor,

$$\mathbf{Y}_L = \frac{1}{\mathbf{Z}_L} = \frac{1}{X_L\angle 90^\circ} = \frac{1}{X_L}\angle -90^\circ$$

Defining $B_L = \frac{1}{X_L}$ (siemens, S)

we have $\mathbf{Y}_L = B_L \angle -90°$

For the capacitor,

$$\mathbf{Y}_C = \frac{1}{\mathbf{Z}_C} = \frac{1}{X_C \angle -90°} = \frac{1}{X_C} \angle 90°$$

Defining $B_C = \frac{1}{X_C}$ (siemens, S)

we have $\mathbf{Y}_C = B_C \angle 90°$

Complex Power

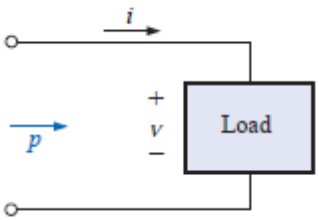
For any system such as in Fig.1, the power delivered to a load at any instant is defined by the product of the applied voltage and the resulting current; that is,

$$p = vi$$

In this case, since *v* and *i* are sinusoidal quantities, let us establish a general case where

$$v = V_m \sin(\omega t + \theta)$$

$$i = I_m \sin \omega t$$



The chosen *v* and *i* include all possibilities because, if the load is resistive, θ = 0°. If the load is purely inductive or capacitive, θ = or θ = - 90°, respectively. For a network that is primarily inductive, θ is positive (*v* leads *i*), and for a network that is primarily capacitive, θ is negative (*i* leads *v*).

Fig.1

Substituting the above equations for *v* and *i* into the power equation will result in

$$p = V_m I_m \sin \omega t \sin(\omega t + \theta)$$

$$\boxed{p = VI \cos \theta(1 - \cos 2\omega t) + VI \sin \theta(\sin 2\omega t)}$$

(1)

where *V* and *I* are the rms values.

$$p = \underbrace{VI \cos \theta}_{\text{Average}} - \underbrace{VI \cos \theta}_{\text{Peak}} \cos \underbrace{2\omega t}_{2x} + \underbrace{VI \sin \theta}_{\text{Peak}} \sin \underbrace{2\omega t}_{2x}$$

- **RESISTIVE CIRCUIT**

For a purely resistive circuit, *v* and *i* are in phase, and θ = 0°,

$$p_R = VI \cos(0°)(1 - \cos 2\omega t) + VI \sin(0°) \sin 2\omega t$$
$$= VI(1 - \cos 2\omega t) + 0$$

$$\boxed{p_R = VI - VI \cos 2\omega t}$$

The total power delivered to a resistor will be dissipated in the form of heat.

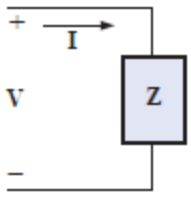
$$\boxed{P = VI = \frac{V_m I_m}{2} = I^2 R = \frac{V^2}{R}} \quad \text{(watts, W)}$$

This is called the real or Active power.

APPARENT POWER

It shown that the power is simply determined by the product of the applied voltage and current, with no concern for the components of the load; that is, $P = VI$. It is called the **apparent power** and is represented symbolically by *S*.*

Since it is simply the product of voltage and current, its units are *voltampere* for which the abbreviation is VA. Its magnitude is determined



$$\boxed{S = VI} \quad \text{(volt-amperes, VA)}$$

or, since $V = IZ$ and $I = \frac{V}{Z}$

then $$\boxed{S = I^2Z} \quad \text{(VA)}$$

and $$\boxed{S = \frac{V^2}{Z}} \quad \text{(VA)}$$

and the power factor of a system *Fp* is

$$P = VI\cos\theta$$

However, $S = VI$

Therefore, $$\boxed{P = S\cos\theta} \quad \text{(W)}$$

and the power factor of a system F_p is

$$\boxed{F_p = \cos\theta = \frac{P}{S}} \quad \text{(unitless)}$$

The power factor of a circuit, therefore, is the ratio of the average power to the apparent power. For a purely resistive circuit, we have

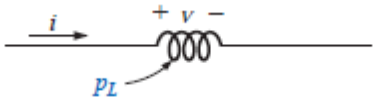
$$P = VI = S$$

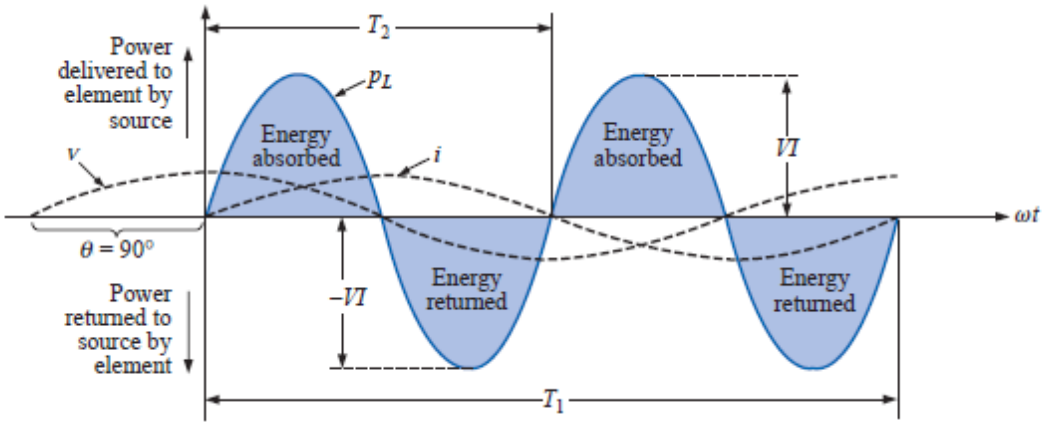
and $$F_p = \cos\theta = \frac{P}{S} = 1$$

INDUCTIVE CIRCUIT AND REACTIVE POWER

For a purely inductive circuit , *v* leads *i* by 90°, as shown in Fig. . Therefore, Substituting θ = 90° in Eq. (1), yields

$$p_L = VI\cos(90°)(1 - \cos 2\omega t) + VI\sin(90°)(\sin 2\omega t)$$
$$= 0 + VI\sin 2\omega t$$





The power curve for a purely inductive load.

or $$p_L = VI \sin 2\omega t$$

- ***The net flow of power to the pure (ideal) inductor is zero over a full cycle, and no energy is lost in the transaction.***

- **In general, the reactive power associated with any circuit is define to be *VI* sin θ, The symbol for reactive power is *Q*, and its unit of measure is the *volt-ampere reactive* (VAR)**

$$Q = VI \sin \theta$$ (volt-ampere reactive, VAR)

where θ is the phase angle between *V* and *I*.
For the inductor,

$$Q_L = VI$$ (VAR)

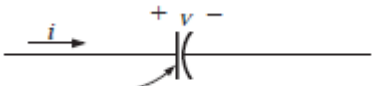
or, since $V = IX_L$ or $I = V/X_L$,

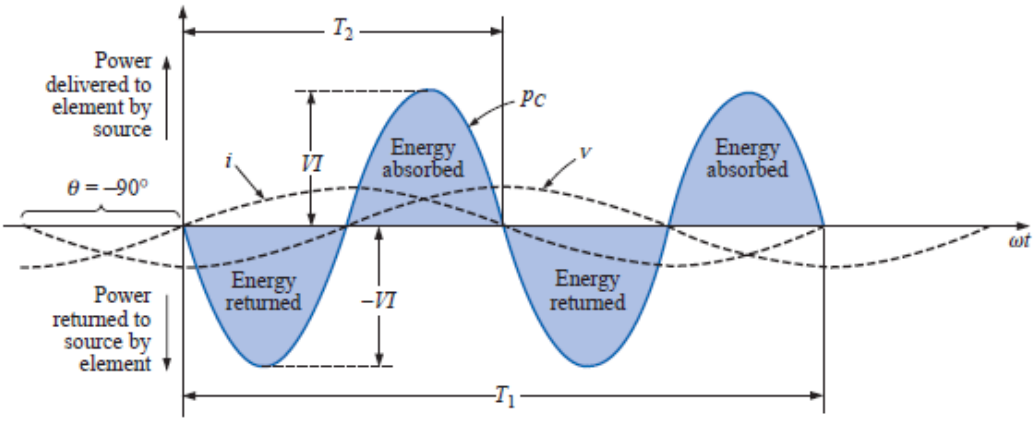
$$Q_L = I^2 X_L$$ (VAR)

or $$Q_L = \frac{V^2}{X_L}$$ (VAR)

CAPACITIVE CIRCUIT

For a purely capacitive circuit , *i* leads *v* by 90°, . Therefore, Substituting θ = -90° , in Eq. (1) we obtain





$$p_C = VI\cos(-90°)(1 - \cos 2\omega t) + VI\sin(-90°)(\sin 2\omega t)$$
$$= 0 - VI\sin 2\omega t$$

or $$\boxed{p_C = -VI\sin 2\omega t}$$ (

- ***The net flow of power to the pure (ideal) capacitor is zero over a full cycle,***

The reactive power associated with the capacitor is equal to the peak value of the *pC* curve, as follows:

$$\boxed{Q_C = VI} \quad \text{(VAR)}$$

But, since $V = IXC$ and $I = V/XC$, the reactive power to the capacitor can also be written

$$\boxed{Q_C = I^2X_C} \quad \text{(VAR)}$$

and $$\boxed{Q_C = \frac{V^2}{X_C}} \quad \text{(VAR)}$$

The apparent power associated with the capacitor is

$$\boxed{S = VI} \quad \text{(VA)}$$

THE POWER TRIANGLE

The three quantities **average power, apparent power,** and **reactive power** can be related in the vector domain by

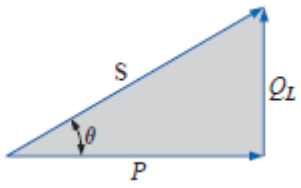
$$\boxed{\mathbf{S} = \mathbf{P} + \mathbf{Q}}$$

with

$$\mathbf{P} = P\angle 0° \qquad \mathbf{Q}_L = Q_L\angle 90° \qquad \mathbf{Q}_C = Q_C\angle -90°$$

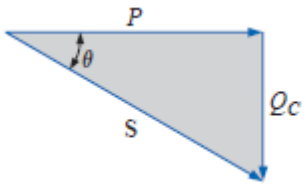
- For an inductive load, the *phasor power* **S,** as it is often called, is defined by

$$\mathbf{S} = P + jQ_L$$

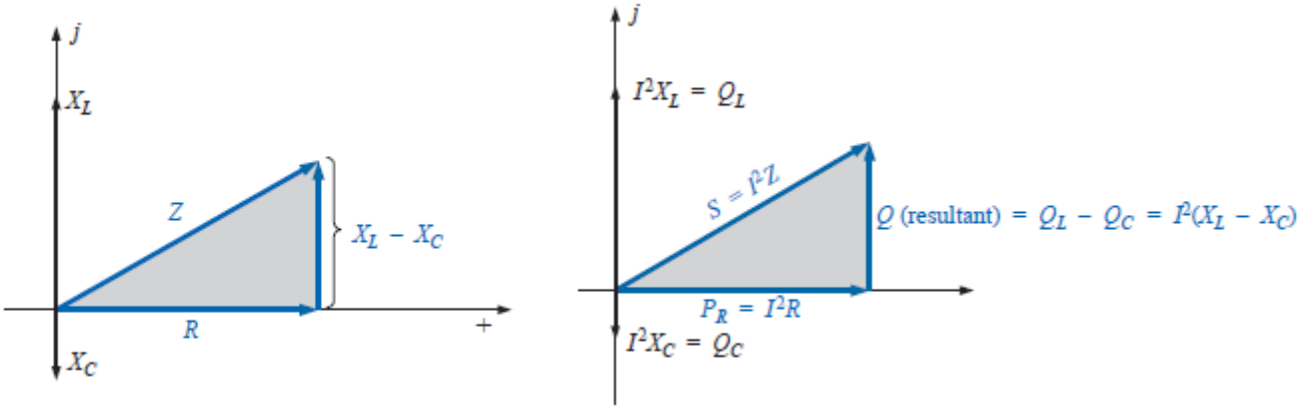


- For a capacitive load, the phasor power **S** is defined by

$$\mathbf{S} = P - jQ_C$$



Complex power For Series RLC circuit

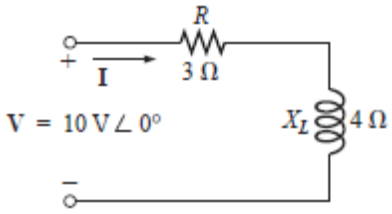


$$S^2 = P^2 + Q^2$$

It is particularly interesting that the equation

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* \qquad (2)$$

will provide the vector form of the apparent power of a system. Here, **V** is the voltage across the system, and **I*** is the complex conjugate of the current.
Consider, for example, the simple *R-L* circuit of Fig. , where



$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_T} = \frac{10\text{ V}\angle 0°}{3\ \Omega + j\,4\ \Omega} = \frac{10\text{ V}\angle 0°}{5\ \Omega\angle 53.13°} = 2\text{ A}\angle -53.13°$$

The real power (the term *real* being derived from the positive real axis of the complex plane) is

$$P = I^2R = (2\text{ A})^2(3\ \Omega) = 12\text{ W}$$

Fig.A

and the reactive power is

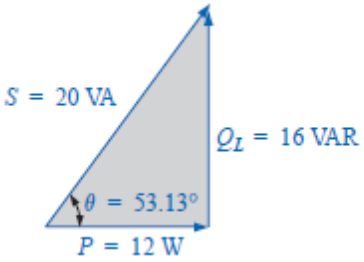
$$Q_L = I^2X_L = (2\text{ A})^2(4\ \Omega) = 16\text{ VAR}\ (L)$$

with $$\mathbf{S} = P + j\,Q_L = 12\text{ W} + j\,16\text{ VAR}\ (L) = 20\text{ VA}\angle 53.13°$$

as shown in Fig. A. Applying Eq. (2) yields

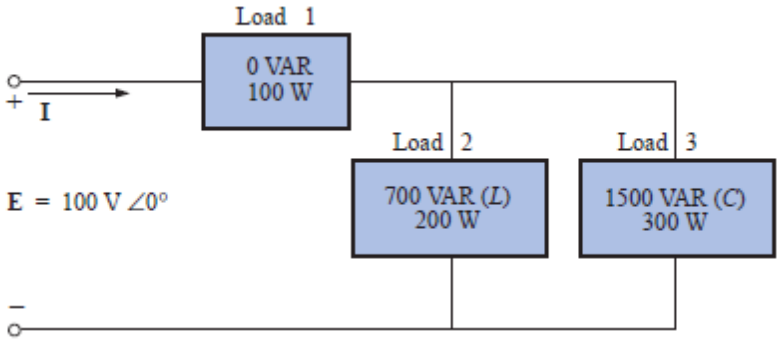
$$\mathbf{S} = \mathbf{VI}^* = (10\,\text{V}\angle 0°)(2\,\text{A}\angle +53.13°) = 20\,\text{VA}\angle 53.13°$$

as obtained above.



EXAMPLE Find the total number of watts, volt-amperes reactive, and volt-amperes, and the power factor *Fp* of the network in Fig.1. Draw the power triangle and find the current in phasor form.

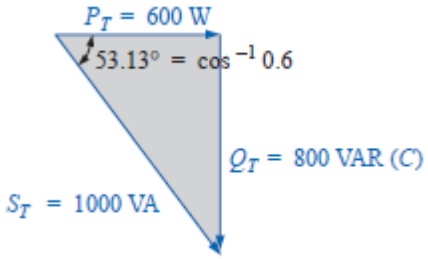
Fig.1



Solution

Load	W	VAR	VA
1	100	0	100
2	200	700 (*L*)	$\sqrt{(200)^2 + (700)^2} = 728.0$
3	300	1500 (*C*)	$\sqrt{(300)^2 + (1500)^2} = 1529.71$
	$P_T = 600$	$Q_T = 800$ (*C*)	$S_T = \sqrt{(600)^2 + (800)^2} = 1000$
	Total power dissipated	Resultant reactive power of network	(Note that $S_T \neq$ sum of each branch: $1000 \neq 100 + 728 + 1529.71$)

$$F_p = \frac{P_T}{S_T} = \frac{600\,\text{W}}{1000\,\text{VA}} = 0.6\ \text{leading}\ (C)$$



The power triangle is shown in Fig. 2.
Since *ST* = *VI* = 1000 VA, *I* = 1000 VA/100 V = 10 A;
and since θ is the angle between the input voltage and current:

$$\mathbf{I} = 10\,\text{A}\angle +53.13°$$

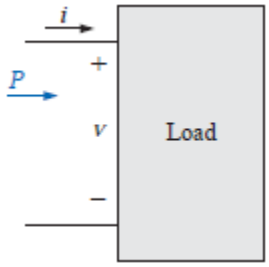
Fig.2

The plus sign is associated with the phase angle since the circuit is predominantly capacitive.

14.5 AVERAGE POWER AND POWER FACTOR

For any load in a sinusoidal ac network, the voltage across the load and the current through the load will vary in a sinusoidal nature. The questions then arise, How does the power to the load determined by the product $v \cdot i$ vary, and what fixed value can be assigned to the power since it will vary with time?
If we take the general case depicted in Fig. 1 and use the following for v and i:

$$v = V_m \sin(\omega t + \theta_v)$$
$$i = I_m \sin(\omega t + \theta_i)$$



then the power is defined by

$$p = vi = V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i)$$
$$= V_m I_m \sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$$

Fig.1

Using the trigonometric identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

the function $\sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$ becomes

$$\sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$$
$$= \frac{\cos[(\omega t + \theta_v) - (\omega t + \theta_i)] - \cos[(\omega t + \theta_v) + (\omega t + \theta_i)]}{2}$$
$$= \frac{\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i)}{2}$$

so that

$$p = \overbrace{\left[\frac{V_m I_m}{2} \cos(\theta_v - \theta_i)\right]}^{\text{Fixed value}} - \overbrace{\left[\frac{V_m I_m}{2} \cos(2\omega t + \theta_v + \theta_i)\right]}^{\text{Time-varying (function of } t\text{)}}$$

A plot of v, i, and p on the same set of axes is shown in Fig. 2. Note that the second factor in the preceding equation is a cosine wave with an amplitude of $V_m I_m/2$ and with a frequency twice that of the voltage or current. The average value of this term is zero over one cycle, producing no net transfer of energy in any one direction.

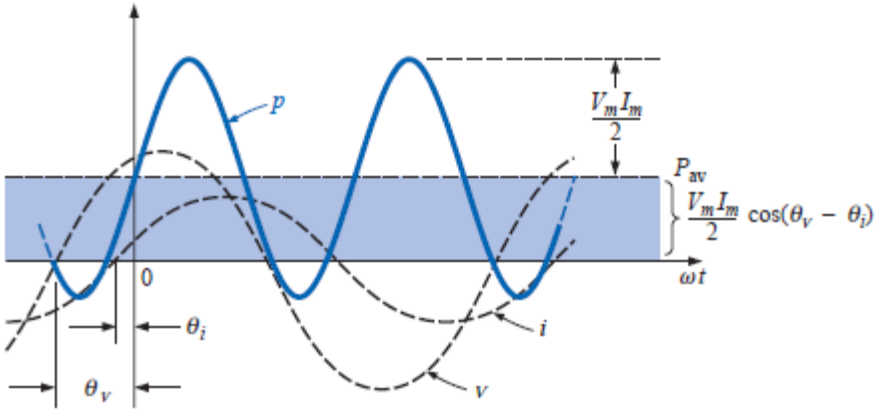


Fig.2

The first term in the preceding equation, however, has a constant magnitude (no time dependence) and therefore provides some net transfer of energy. This term is referred to as the **average power,** the reason for which is obvious from Fig. 2. The average power, or **real power** as it is sometimes called, is the power delivered to and dissipated by the load. It corresponds to the power calculations performed for dc networks. The angle ($\theta_v - \theta_i$) is the phase angle between *v* and *i*. Since cos(-a) = cos a,

the magnitude of average power delivered is independent of whether v leads i or i leads v.

Defining as equal to $|\theta_v - \theta_i|$, where | |indicates that only the magnitude is important and the sign is immaterial, we have

$$P = \frac{V_m I_m}{2} \cos\theta \qquad \text{(watts, W)}$$

where *P* is the average power in watts. This equation can also be written

$$P = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

or, since $V_{\text{eff}} = \frac{V_m}{\sqrt{2}}$ and $I_{\text{eff}} = \frac{I_m}{\sqrt{2}}$

Hence $$P = V_{\text{eff}} I_{\text{eff}} \cos\theta$$

- ## Resistor

In a purely resistive circuit, since *v* and *i* are in phase, $|\theta_v - \theta_i| = \theta = 0°$, and $\cos\theta = \cos 0° = 1$, so that

$$P = \frac{V_m I_m}{2} = V_{\text{eff}} I_{\text{eff}} \qquad \text{(W)}$$

Or, since $I_{\text{eff}} = \frac{V_{\text{eff}}}{R}$

then $$P = \frac{V_{\text{eff}}^2}{R} = I_{\text{eff}}^2 R \qquad \text{(W)}$$

- ## Inductor

In a purely inductive circuit, since *v* leads *i* by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos 90° = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal inductor (no associated resistance) is zero watts.

- **Capacitor**

In a purely capacitive circuit, since *i* leads *v* by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2}\cos(90°) = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal capacitor (no associated resistance) is zero watts.

EXAMPLE 1 Find the average power dissipated in a network whose input current and voltage are the following:

$i = 5\sin(\omega t + 40°)$

$v = 10\sin(\omega t + 40°)$

Solution: Since *v* and *i* are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

$$P = \frac{V_m I_m}{2} = \frac{(10\text{ V})(5\text{ A})}{2} = \mathbf{25\ W}$$

or

$$R = \frac{V_m}{I_m} = \frac{10\text{ V}}{5\text{ A}} = 2\ \Omega$$

and

$$P = \frac{V_{\text{eff}}^2}{R} = \frac{[(0.707)(10\text{ V})]^2}{2} = \mathbf{25\ W}$$

or

$$P = I_{\text{eff}}^2 R = [(0.707)(5\text{ A})]^2(2) = \mathbf{25\ W}$$

EXAMPLE 2: Determine the average power delivered to networks having the following input voltage and current:

a. $v = 100\sin(\omega t + 40°)$
   $i = 20\sin(\omega t + 70°)$
b. $v = 150\sin(\omega t - 70°)$
   $i = 3\sin(\omega t - 50°)$

Solutions:

a. $V_m = 100, \quad \theta_v = 40°$
   $I_m = 20, \quad \theta_i = 70°$
   $\theta = |\theta_v - \theta_i| = |40° - 70°| = |-30°| = 30°$
   and

$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(100\text{ V})(20\text{ A})}{2}\cos(30°) = (1000\text{ W})(0.866)$$
$$= \mathbf{866\ W}$$

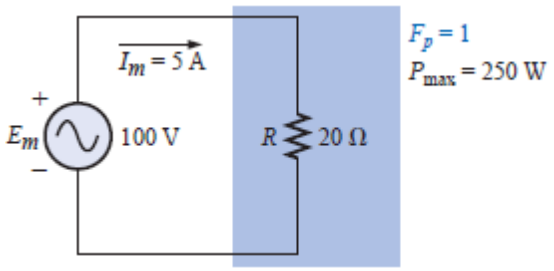
b. $V_m = 150\text{ V}, \quad \theta_v = -70°$
$I_m = 3\text{ A}, \quad \theta_i = -50°$
$\theta = |\theta_v - \theta_i| = |-70° - (-50°)|$
$= |-70° + 50°| = |-20°| = 20°$
and

$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(150\text{ V})(3\text{ A})}{2}\cos(20°) = (225\text{ W})(0.9397)$$
$$= \mathbf{211.43\text{ W}}$$

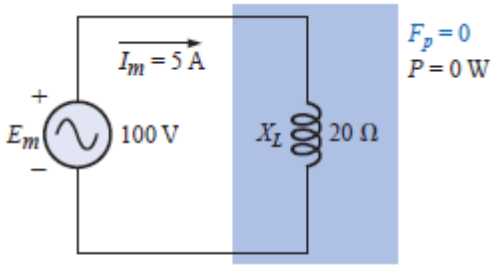
Power Factor

In the equation $P = (V_m I_m/2)\cos\theta$, the factor that has significant control over the delivered power level is the $\cos\theta$. No matter how large the voltage or current, if $\cos\theta = 0$, the power is zero; if $\cos\theta = 1$, the power delivered is a maximum. Since it has such control, the expression was given the name **power factor** and is defined by

$$\text{Power factor} = F_p = \cos\theta$$



Purely resistive load with $F_p = 1$.



Purely inductive load with $F_p = 0$.

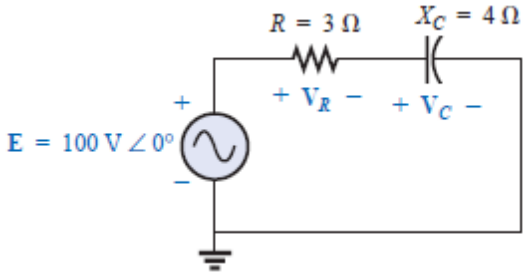
15.4 VOLTAGE DIVIDER RULE

The basic format for the **voltage divider rule** in ac circuits is exactly the same as that for dc circuits:

$$\mathbf{V}_x = \frac{\mathbf{Z}_x\mathbf{E}}{\mathbf{Z}_T}$$

where $\mathbf{V}_x$ is the voltage across one or more elements in series that have total impedance $\mathbf{Z}_x$, $\mathbf{E}$ is the total voltage appearing across the series circuit, and $\mathbf{Z}_T$ is the total impedance of the series circuit.

EXAMPLE 15.9 Using the voltage divider rule, find the voltage across each element of the circuit of Fig. 15.40.



Solution:

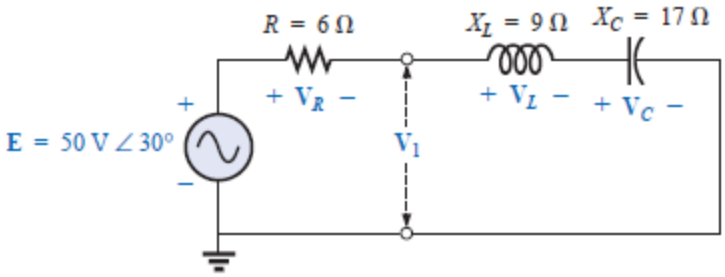
$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(4\ \Omega \angle -90°)(100\ \text{V} \angle 0°)}{4\ \Omega \angle -90° + 3\ \Omega \angle 0°} = \frac{400 \angle -90°}{3 - j\,4}$$

$$= \frac{400 \angle -90°}{5 \angle -53.13°} = \mathbf{80\ V \angle -36.87°}$$

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(3\ \Omega \angle 0°)(100\ \text{V} \angle 0°)}{5\ \Omega \angle -53.13°} = \frac{300 \angle 0°}{5 \angle -53.13°}$$

$$= \mathbf{60\ V \angle +53.13°}$$

EXAMPLE 15.10 Using the voltage divider rule, find the unknown voltages $\mathbf{V}_R$, $\mathbf{V}_L$, $\mathbf{V}_C$, and $\mathbf{V}_1$ for the circuit of Fig. 15.41.



Solution:

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C} = \frac{(6\ \Omega \angle 0°)(50\ \text{V} \angle 30°)}{6\ \Omega \angle 0° + 9\ \Omega \angle 90° + 17\ \Omega \angle -90°}$$

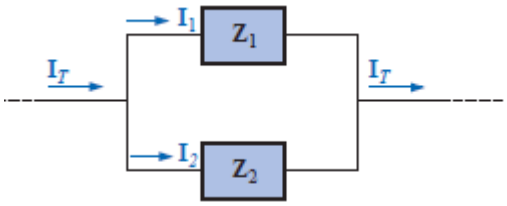
$$= \frac{300 \angle 30°}{6 + j\,9 - j\,17} = \frac{300 \angle 30°}{6 - j\,8}$$

$$= \frac{300 \angle 30°}{10 \angle -53.13°} = \mathbf{30\ V \angle 83.13°}$$

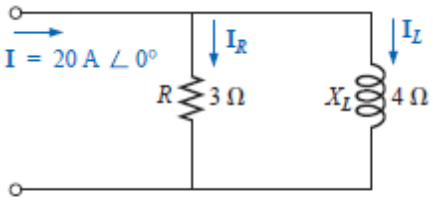
9 CURRENT DIVIDER RULE

The basic format for the **current divider rule** in ac circuits is exactly the same as that for dc circuits; that is, for two parallel branches with impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ as shown in Fig. 15.76,

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2\mathbf{I}_T}{\mathbf{Z}_1 + \mathbf{Z}_2} \quad \text{or} \quad \mathbf{I}_2 = \frac{\mathbf{Z}_1\mathbf{I}_T}{\mathbf{Z}_1 + \mathbf{Z}_2}$$



EXAMPLE 15.15 Using the current divider rule, find the current through each impedance of Fig. 15.77.

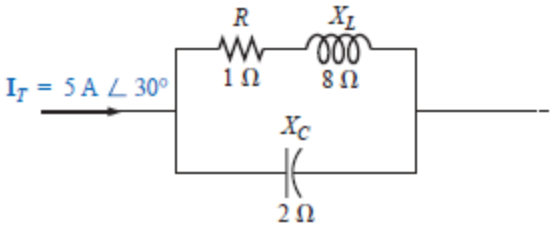


Solution:

$$\mathbf{I}_R = \frac{\mathbf{Z}_L\mathbf{I}_T}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(4\ \Omega\ \angle 90°)(20\ \text{A}\ \angle 0°)}{3\ \Omega\ \angle 0° + 4\ \Omega\ \angle 90°} = \frac{80\ \text{A}\angle 90°}{5\ \angle 53.13°}$$
$$= \mathbf{16\ A\ \angle 36.87°}$$

$$\mathbf{I}_L = \frac{\mathbf{Z}_R\mathbf{I}_T}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(3\ \Omega\ \angle 0°)(20\ \text{A}\ \angle 0°)}{5\ \Omega\ \angle 53.13°} = \frac{60\ \text{A}\angle 0°}{5\ \angle 53.13°}$$
$$= \mathbf{12\ A\ \angle -53.13°}$$

EXAMPLE 15.16 Using the current divider rule, find the current through each parallel branch of Fig. 15.78.



Solution:

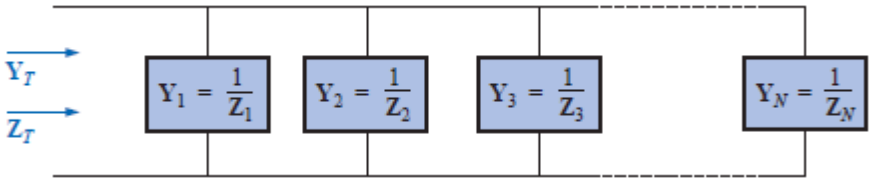
$$\mathbf{I}_{R\text{-}L} = \frac{\mathbf{Z}_C\mathbf{I}_T}{\mathbf{Z}_C + \mathbf{Z}_{R\text{-}L}} = \frac{(2\ \Omega\ \angle -90°)(5\ \text{A}\ \angle 30°)}{-j\,2\ \Omega + 1\ \Omega + j\,8\ \Omega} = \frac{10\ \text{A}\angle -60°}{1 + j\,6}$$
$$= \frac{10\ \text{A}\angle -60°}{6.083\ \angle 80.54°} \cong \mathbf{1.644\ A\ \angle -140.54°}$$

$$\mathbf{I}_C = \frac{\mathbf{Z}_{R\text{-}L}\mathbf{I}_T}{\mathbf{Z}_{R\text{-}L} + \mathbf{Z}_C} = \frac{(1\ \Omega + j\,8\ \Omega)(5\ \text{A}\ \angle 30°)}{6.08\ \Omega\ \angle 80.54°}$$
$$= \frac{(8.06\ \angle 82.87°)(5\ \text{A}\angle 30°)}{6.08\ \angle 80.54°} = \frac{40.30\ \text{A}\angle 112.87°}{6.083\ \angle 80.54°}$$
$$= \mathbf{6.625\ A\ \angle 32.33°}$$

15.7 ADMITTANCE AND SUSCEPTANCE

The discussion for **parallel ac circuits** will be very similar to that for dc circuits. In dc circuits, *conductance* (*G*) was defined as being equal to 1/*R*. The total conductance of a parallel circuit was then found by adding the conductance of each branch. The total resistance R_T is simply $1/G_T$.
In ac circuits, we define **admittance** (**Y**) as being equal to 1/**Z**. The unit of measure for admittance as defined by the SI system is *siemens*, which has the symbol S. Admittance is a measure of how well an ac circuit will *admit*, or allow, current to flow in the circuit. The larger its value, therefore, the heavier the current flow for the same applied potential. The total admittance of a circuit can also be found by finding the sum of the parallel admittances. The total impedance $\mathbf{Z}_T$ of the circuit is then $1/\mathbf{Y}_T$; that is, for the network of Fig. 15.54:

$$\mathbf{Y}_T = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \cdots + \mathbf{Y}_N$$



or, since **Z** =1/**Y**,

$$\frac{1}{\mathbf{Z}_T} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots + \frac{1}{\mathbf{Z}_N}$$

For two impedances in parallel,

$$\frac{1}{\mathbf{Z}_T} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

or

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

As pointed out in the introduction to this section, conductance is the reciprocal of resistance, and

$$\mathbf{Y}_R = \frac{1}{\mathbf{Z}_R} = \frac{1}{R\angle 0°} = G\angle 0°$$

The reciprocal of reactance (1/*X*) is called **susceptance** and is a measure of how *susceptible* an element is to the passage of current through it. Susceptance is also measured in *siemens* and is represented by the capital letter *B*.
For the inductor,

$$\mathbf{Y}_L = \frac{1}{\mathbf{Z}_L} = \frac{1}{X_L\angle 90°} = \frac{1}{X_L}\angle -90°$$

Defining $B_L = \frac{1}{X_L}$ (siemens, S)

we have $\mathbf{Y}_L = B_L\angle -90°$

For the capacitor,

$$\mathbf{Y}_C = \frac{1}{\mathbf{Z}_C} = \frac{1}{X_C\angle -90°} = \frac{1}{X_C}\angle 90°$$

Defining $B_C = \frac{1}{X_C}$ (siemens, S)

we have $\mathbf{Y}_C = B_C\angle 90°$

Complex Power

For any system such as in Fig.1, the power delivered to a load at any instant is defined by the product of the applied voltage and the resulting current; that is,

$$p = vi$$

In this case, since *v* and *i* are sinusoidal quantities, let us establish a general case where

$$v = V_m \sin(\omega t + \theta)$$
$$i = I_m \sin \omega t$$

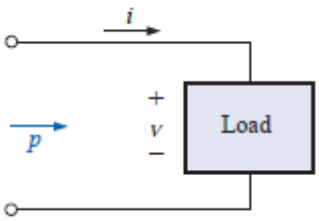


Fig.1

The chosen *v* and *i* include all possibilities because, if the load is resistive, θ = 0°. If the load is purely inductive or capacitive, θ = or θ = - 90°, respectively. For a network that is primarily inductive, θ is positive (*v* leads *i*), and for a network that is primarily capacitive, θ is negative (*i* leads *v*).

Substituting the above equations for *v* and *i* into the power equation will result in

$$p = V_m I_m \sin \omega t \sin(\omega t + \theta)$$

$$\boxed{p = VI \cos \theta(1 - \cos 2\omega t) + VI \sin \theta(\sin 2\omega t)} \quad (1)$$

where *V* and *I* are the rms values.

$$p = \underbrace{VI \cos \theta}_{\text{Average}} - \underbrace{VI \cos \theta \cos 2\omega t}_{\text{Peak}\ \ 2x} + \underbrace{VI \sin \theta \sin 2\omega t}_{\text{Peak}\ \ 2x}$$

- **RESISTIVE CIRCUIT**

For a purely resistive circuit, *v* and *i* are in phase, and θ = 0°,

$$p_R = VI \cos(0°)(1 - \cos 2\omega t) + VI \sin(0°) \sin 2\omega t$$
$$= VI(1 - \cos 2\omega t) + 0$$

$$\boxed{p_R = VI - VI \cos 2\omega t}$$

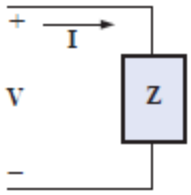
The total power delivered to a resistor will be dissipated in the form of heat.

$$P = VI = \frac{V_m I_m}{2} = I^2R = \frac{V^2}{R} \quad \text{(watts, W)}$$

This is called the real or Active power.

APPARENT POWER

It shown that the power is simply determined by the product of the applied voltage and current, with no concern for the components of the load; that is, $P = VI$. It is called the **apparent power** and is represented symbolically by S.*
Since it is simply the product of voltage and current, its units are *voltampere* for which the abbreviation is VA. Its magnitude is determined



$$S = VI \quad \text{(volt-amperes, VA)}$$

or, since $V = IZ$ and $I = \frac{V}{Z}$

then $$S = I^2Z \quad \text{(VA)}$$

and $$S = \frac{V^2}{Z} \quad \text{(VA)}$$

and the power factor of a system Fp is

$$P = VI \cos\theta$$

However, $$S = VI$$

Therefore, $$P = S \cos\theta \quad \text{(W)}$$

and the power factor of a system F_p is

$$F_p = \cos\theta = \frac{P}{S} \quad \text{(unitless)}$$

The power factor of a circuit, therefore, is the ratio of the average power to the apparent power. For a purely resistive circuit, we have

$$P = VI = S$$

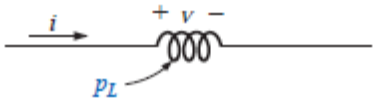
and

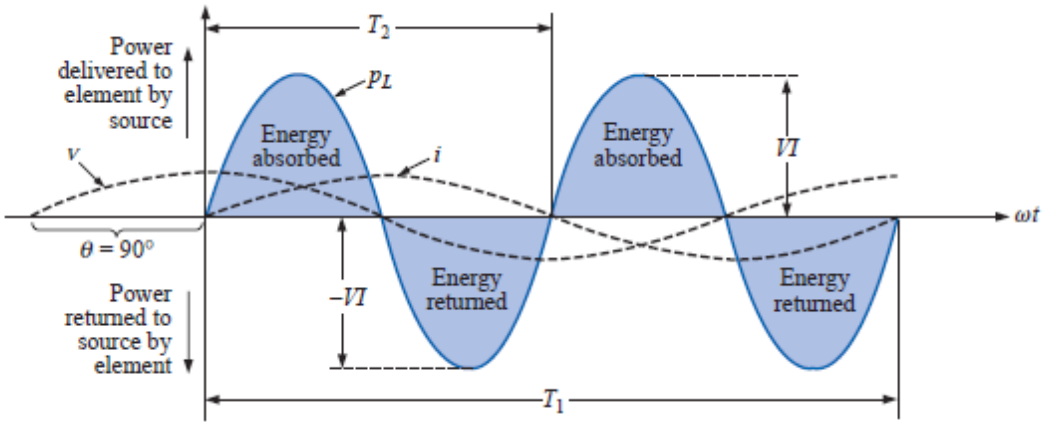
$$F_p = \cos\theta = \frac{P}{S} = 1$$

INDUCTIVE CIRCUIT AND REACTIVE POWER

For a purely inductive circuit , *v* leads *i* by 90°, as shown in Fig. . Therefore, Substituting θ = 90° in Eq. (1), yields

$$p_L = VI\cos(90°)(1 - \cos 2\omega t) + VI\sin(90°)(\sin 2\omega t)$$
$$= 0 + VI\sin 2\omega t$$





The power curve for a purely inductive load.

or

$$p_L = VI\sin 2\omega t$$

- ***The net flow of power to the pure (ideal) inductor is zero over a full cycle, and no energy is lost in the transaction.***

- **In general, the reactive power associated with any circuit is define to be *VI* sin θ, The symbol for reactive power is *Q*, and its unit of measure is the *volt-ampere reactive* (VAR)**

$$Q = VI \sin\theta \quad \text{(volt-ampere reactive, VAR)}$$

where θ is the phase angle between *V* and *I*.
For the inductor,

$$Q_L = VI \quad \text{(VAR)}$$

or, since $V = IX_L$ or $I = V/X_L$,

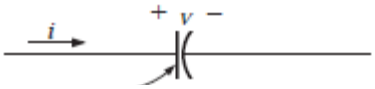
$$Q_L = I^2X_L \quad \text{(VAR)}$$

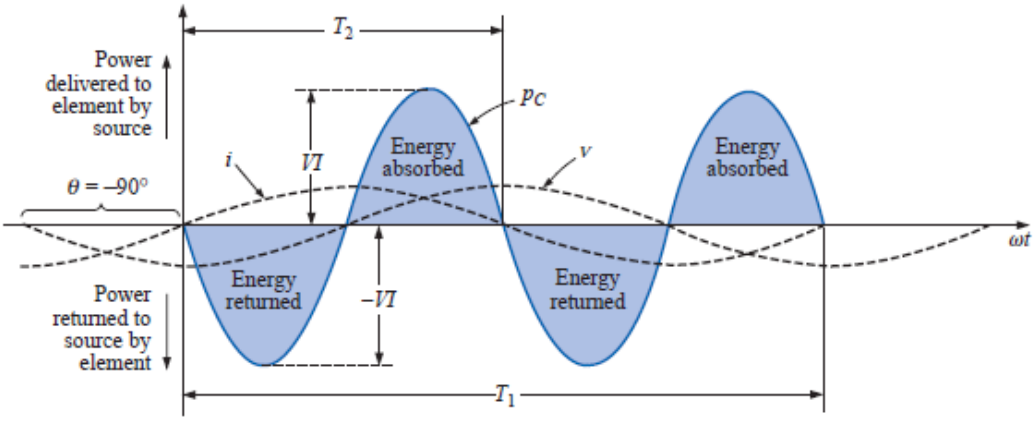
or

$$Q_L = \frac{V^2}{X_L} \quad \text{(VAR)}$$

CAPACITIVE CIRCUIT

For a purely capacitive circuit , *i* leads *v* by 90°, . Therefore, Substituting θ = -90° , in Eq. (1) we obtain





$$p_C = VI\cos(-90°)(1 - \cos 2\omega t) + VI\sin(-90°)(\sin 2\omega t)$$
$$= 0 - VI\sin 2\omega t$$

or

$$p_C = -VI\sin 2\omega t \quad ($$

- ***The net flow of power to the pure (ideal) capacitor is zero over a full cycle,***

The reactive power associated with the capacitor is equal to the peak value of the *pC* curve, as follows:

$$\boxed{Q_C = VI} \quad \text{(VAR)}$$

But, since $V = IXC$ and $I = V/XC$, the reactive power to the capacitor can also be written

$$\boxed{Q_C = I^2 X_C} \quad \text{(VAR)}$$

and

$$\boxed{Q_C = \frac{V^2}{X_C}} \quad \text{(VAR)}$$

The apparent power associated with the capacitor is

$$\boxed{S = VI} \quad \text{(VA)}$$

THE POWER TRIANGLE

The three quantities **average power, apparent power,** and **reactive power** can be related in the vector domain by

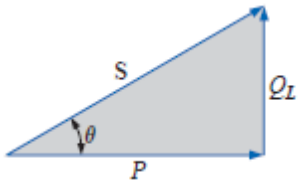
$$\boxed{\mathbf{S} = \mathbf{P} + \mathbf{Q}}$$

with

$$\mathbf{P} = P\angle 0° \qquad \mathbf{Q}_L = Q_L\angle 90° \qquad \mathbf{Q}_C = Q_C\angle -90°$$

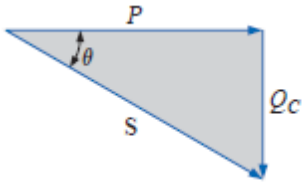
- For an inductive load, the *phasor power* **S,** as it is often called, is defined by

$$\mathbf{S} = P + j\,Q_L$$

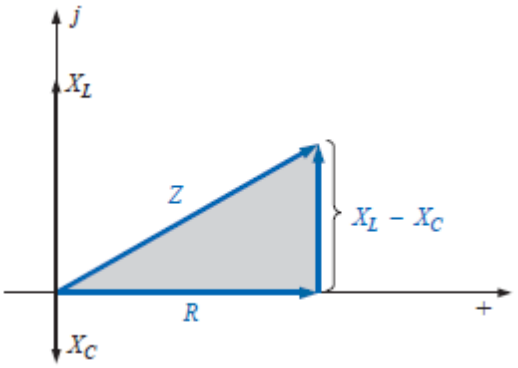


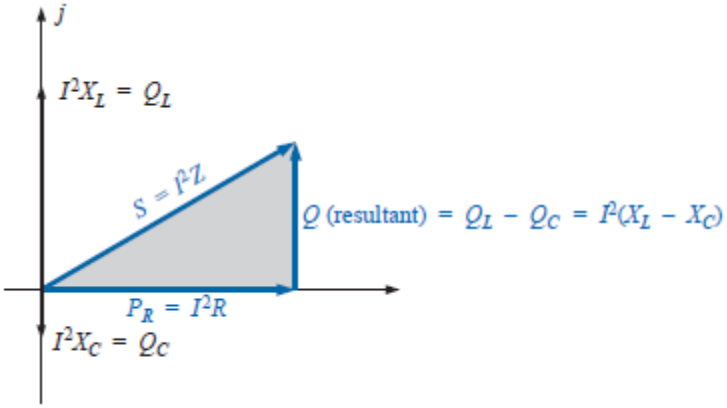
- For a capacitive load, the phasor power **S** is defined by

$$\mathbf{S} = P - j\,Q_C$$



Complex power For Series RLC circuit



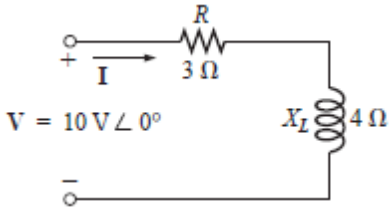


$$S^2 = P^2 + Q^2$$

It is particularly interesting that the equation

$$\mathbf{S} = \mathbf{V}\mathbf{I}^*$$

will provide the vector form of the apparent power of a system. Here, **V** is the voltage across the system, and **I*** is the complex conjugate of the current.
Consider, for example, the simple *R-L* circuit of Fig. , where



$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_T} = \frac{10\text{ V}\angle 0°}{3\ \Omega + j\,4\ \Omega} = \frac{10\text{ V}\angle 0°}{5\ \Omega\angle 53.13°} = 2\text{ A}\angle -53.13°$$

The real power (the term *real* being derived from the positive real axis of the complex plane) is

$$P = I^2R = (2\text{ A})^2(3\ \Omega) = 12\text{ W}$$

and the reactive power is

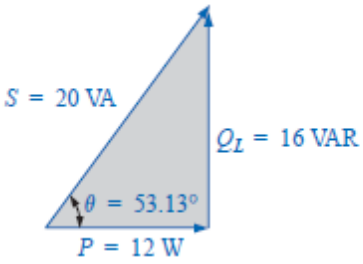
$$Q_L = I^2X_L = (2\text{ A})^2(4\ \Omega) = 16\text{ VAR}\ (L)$$

with $$\mathbf{S} = P + j\,Q_L = 12\text{ W} + j\,16\text{ VAR}\ (L) = 20\text{ VA}\angle 53.13°$$

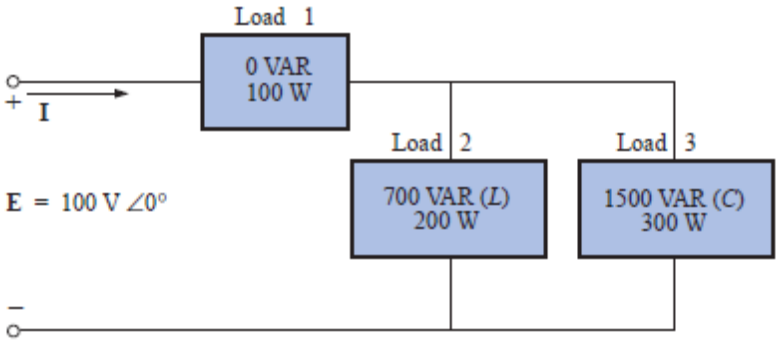
as shown in Fig. 19.16. Applying Eq. (19.29) yields

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = (10\text{ V}\angle 0°)(2\text{ A}\angle +53.13°) = 20\text{ VA}\angle 53.13°$$

as obtained above.



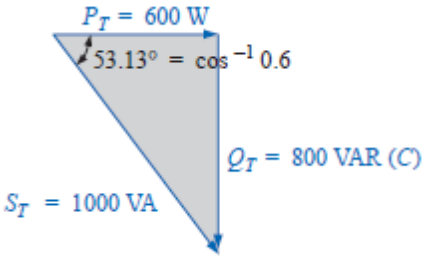
EXAMPLE 19.1 Find the total number of watts, volt-amperes reactive, and volt-amperes, and the power factor *Fp* of the network in Fig.17. Draw the power triangle and find the current in phasor form.



Solution

Load	W	VAR	VA
1	100	0	100
2	200	700 (*L*)	$\sqrt{(200)^2 + (700)^2} = 728.0$
3	300	1500 (*C*)	$\sqrt{(300)^2 + (1500)^2} = 1529.71$
	$P_T = 600$ Total power dissipated	$Q_T = 800$ (*C*) Resultant reactive power of network	$S_T = \sqrt{(600)^2 + (800)^2} = 1000$ (Note that $S_T \neq$ sum of each branch: $1000 \neq 100 + 728 + 1529.71$)

$$F_p = \frac{P_T}{S_T} = \frac{600 \text{ W}}{1000 \text{ VA}} = \textbf{0.6 leading (}\boldsymbol{C}\textbf{)}$$



The power triangle is shown in Fig. 19.18.
Since *ST* = *VI* = 1000 VA, *I* = 1000 VA/100 V = 10 A; and
of cos θ = *Fp* is the angle between the input voltage and cu

$$\mathbf{I} = \mathbf{10\ A} \angle \mathbf{+53.13^\circ}$$

The plus sign is associated with the phase angle since the circuit is predominantly capacitive.

Δ-Y, Y- Δ CONVERSIONS

The Δ-Y, Y- Δ conversions for ac circuits will not be derived here since the development corresponds exactly with that for dc circuits. Taking the **Δ -Y configuration** shown in Fig. 1, we find the general equations for the impedances of the Y in terms of those for the Δ :

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_B\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} \quad (1)$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_A\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} \quad (2)$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_A\mathbf{Z}_B}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} \quad (3)$$

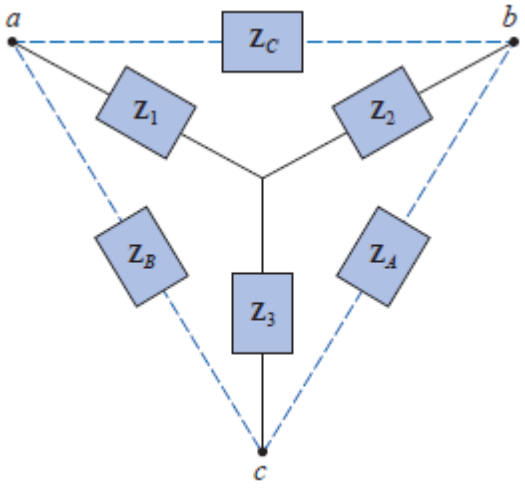


Fig.1 **Δ -Y configuration**

For the impedances of the Δ in terms of those for the Y, the equations are

$$\mathbf{Z}_B = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2} \quad (4)$$

$$\mathbf{Z}_A = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1} \quad (5)$$

$$\mathbf{Z}_C = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_3} \quad (6)$$

- ***Note that each impedance of the Y is equal to the product of the impedances in the two closest branches of the Δ , divided by the sum of the impedances in the Δ .***
- ***Further, the value of each impedance of the Δ is equal to the sum of the possible product combinations of the impedances of the Y, divided by the impedances of theYfarthest from the impedance to be determined.***

EXAMPLE 17.20 Find the total impedance $\mathbf{Z}_T$ of the network of Fig.

EXAMPLE 1 Find the total impedance $\mathbf{Z}_T$ of the network of Fig.2

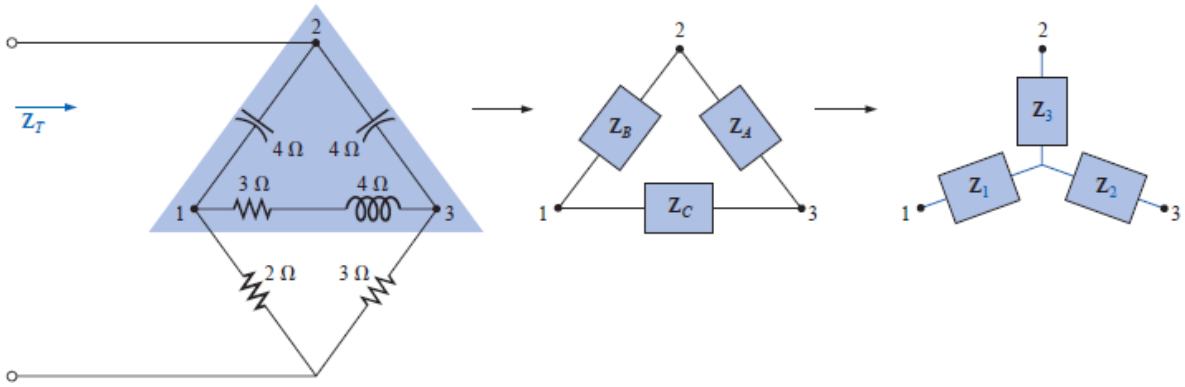


Fig.1 *Converting the upper Δ of a bridge configuration to a Y.*

Solution

$$\mathbf{Z}_B = -j\,4 \qquad \mathbf{Z}_A = -j\,4 \qquad \mathbf{Z}_C = 3 + j\,4$$

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_B\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} = \frac{(-j\,4\ \Omega)(3\ \Omega + j\,4\ \Omega)}{(-j\,4\ \Omega) + (-j\,4\ \Omega) + (3\ \Omega + j\,4\ \Omega)}$$

$$= \frac{(4\ \angle -90°)(5\ \angle 53.13°)}{3 - j\,4} = \frac{20\ \angle -36.87°}{5\ \angle -53.13°}$$

$$= 4\ \Omega\ \angle 16.13° = 3.84\ \Omega + j\ 1.11\ \Omega$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_A\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} = \frac{(-j\,4\ \Omega)(3\ \Omega + j\,4\ \Omega)}{5\ \Omega\ \angle -53.13°}$$

$$= 4\ \Omega\ \angle 16.13° = 3.84\ \Omega + j\ 1.11\ \Omega$$

Recall from the study of dc circuits that if two branches of the Y or Δ were the same, the corresponding Δ or Y, respectively, would also have two similar branches. In this example, $\mathbf{Z}_A = \mathbf{Z}_B$. Therefore, $\mathbf{Z}_1 = \mathbf{Z}_2$,

And

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_A\mathbf{Z}_B}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} = \frac{(-j\,4\ \Omega)(-j\,4\ \Omega)}{5\ \Omega\angle -53.13^\circ}$$
$$= \frac{16\ \Omega\angle -180^\circ}{5\angle -53.13^\circ} = 3.2\ \Omega\angle -126.87^\circ = -1.92\ \Omega - j\,2.56\ \Omega$$

Replace the Δ by the Y (Fig. 2):

$\mathbf{Z}_1 = 3.84\ \Omega + j\,1.11\ \Omega$ $\mathbf{Z}_2 = 3.84\ \Omega + j\,1.11\ \Omega$
$\mathbf{Z}_3 = -1.92\ \Omega$
$\mathbf{Z}_5 = 3\ \Omega$

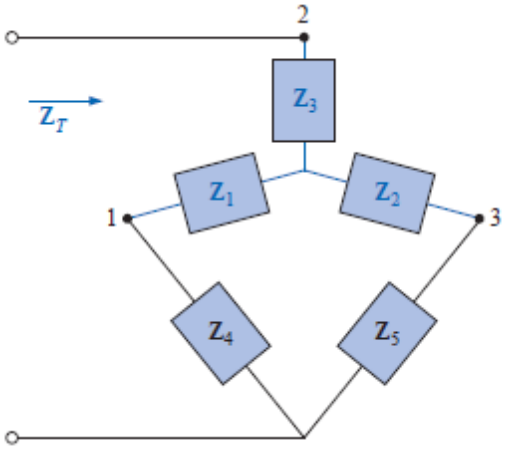


Fig.2 *The network of Fig. 1 following the substitution of the Y configuration.*

Impedances $\mathbf{Z}_1$ and $\mathbf{Z}_4$ are in series:

$$\mathbf{Z}_{T_1} = \mathbf{Z}_1 + \mathbf{Z}_4 = 3.84\ \Omega + j\,1.11\ \Omega + 2\ \Omega = 5.84\ \Omega + j\,1.11\ \Omega$$
$$= 5.94\ \Omega\angle 10.76^\circ$$

Impedances $\mathbf{Z}_2$ and $\mathbf{Z}_5$ are in series:

$$\mathbf{Z}_{T_2} = \mathbf{Z}_2 + \mathbf{Z}_5 = 3.84\ \Omega + j\,1.11\ \Omega + 3\ \Omega = 6.84\ \Omega + j\,1.11\ \Omega$$
$$= 6.93\ \Omega\angle 9.22^\circ$$

Impedances $\mathbf{Z}_{T_1}$ and $\mathbf{Z}_{T_2}$ are in parallel:

$$\mathbf{Z}_{T_3} = \frac{\mathbf{Z}_{T_1}\mathbf{Z}_{T_2}}{\mathbf{Z}_{T_1} + \mathbf{Z}_{T_2}} = \frac{(5.94\ \Omega\angle 10.76^\circ)(6.93\ \Omega\angle 9.22^\circ)}{5.84\ \Omega + j\,1.11\ \Omega + 6.84\ \Omega + j\,1.11\ \Omega}$$
$$= \frac{41.16\ \Omega\angle 19.98^\circ}{12.68 + j\,2.22} = \frac{41.16\ \Omega\angle 19.98^\circ}{12.87\angle 9.93^\circ} = 3.198\ \Omega\angle 10.05^\circ$$
$$= 3.15\ \Omega + j\,0.56\ \Omega$$

Impedances $\mathbf{Z}_3$ and $\mathbf{Z}_{T_3}$ are in series. Therefore,

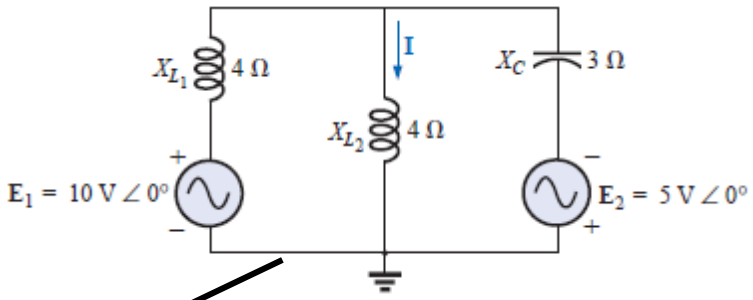
$$\mathbf{Z}_T = \mathbf{Z}_3 + \mathbf{Z}_{T_3} = -1.92\ \Omega - j\,2.56\ \Omega + 3.15\ \Omega + j\,0.56\ \Omega$$
$$= 1.23\ \Omega - j\,2.0\ \Omega = \mathbf{2.35\ \Omega\angle -58.41^\circ}$$

Network Theorems (ac)

1- SUPERPOSITION THEOREM

EXAMPLE 1 Using the superposition theorem, find the current **I** through the 4-Ω reactance (X_{L2}) of Fig. 1.

Fig.1



Solution

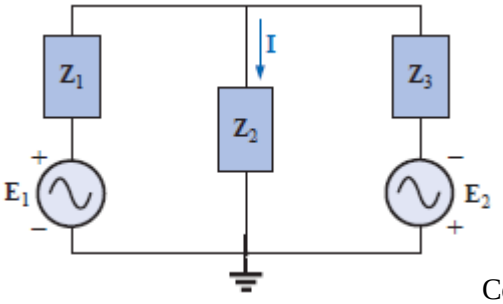


Fig.2

For the re-drawn circuit in Fig.2

$$\mathbf{Z}_1 = +jX_{L_1} = j\,4\ \Omega$$
$$\mathbf{Z}_2 = +jX_{L_2} = j\,4\ \Omega$$
$$\mathbf{Z}_3 = -jX_C = -j\,3\ \Omega$$

Considering the effects of the voltage source **E**1 (Fig. 3), we have

$$\mathbf{Z}_{2\|3} = \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2 + \mathbf{Z}_3} = \frac{(j\,4\ \Omega)(-j\,3\ \Omega)}{j\,4\ \Omega - j\,3\ \Omega} = \frac{12\ \Omega}{j} = -j\,12\ \Omega$$
$$= 12\ \Omega \angle -90°$$
$$I_{s_1} = \frac{\mathbf{E}_1}{\mathbf{Z}_{2\|3} + \mathbf{Z}_1} = \frac{10\ \text{V} \angle 0°}{-j\,12\ \Omega + j\,4\ \Omega} = \frac{10\ \text{V} \angle 0°}{8\ \Omega \angle -90°}$$
$$= 1.25\ \text{A} \angle 90°$$
$$\mathbf{I}' = \frac{\mathbf{Z}_3\mathbf{I}_{s_1}}{\mathbf{Z}_2 + \mathbf{Z}_3} \quad \text{(current divider rule)}$$
$$= \frac{(-j\,3\ \Omega)(j\,1.25\ \text{A})}{j\,4\ \Omega - j\,3\ \Omega} = \frac{3.75\ \text{A}}{j\,1} = 3.75\ \text{A} \angle -90°$$

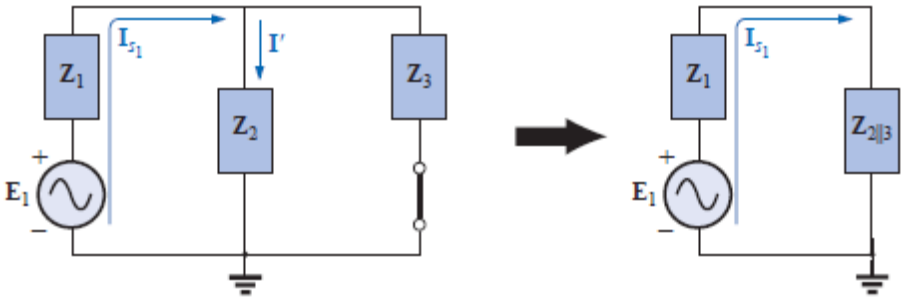
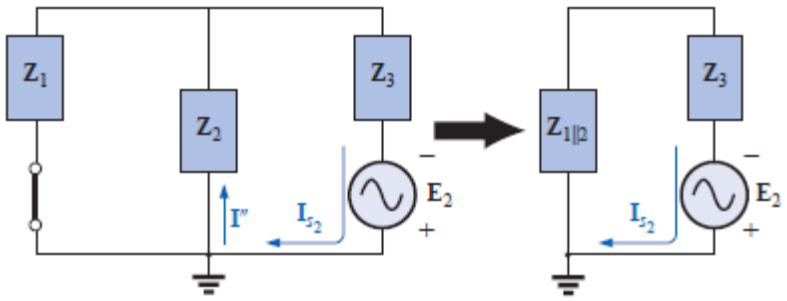


Fig.3

Fig.4



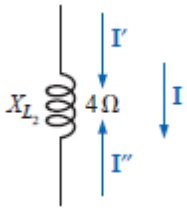
Considering the effects of the voltage source **E**2 (Fig.4), we have

$$\mathbf{Z}_{1\|2} = \frac{\mathbf{Z}_1}{N} = \frac{j\,4\ \Omega}{2} = j\,2\ \Omega$$

$$\mathbf{I}_{s_2} = \frac{\mathbf{E}_2}{\mathbf{Z}_{1\|2} + \mathbf{Z}_3} = \frac{5\ \text{V}\ \angle 0°}{j\,2\ \Omega - j\,3\ \Omega} = \frac{5\ \text{V}\ \angle 0°}{1\ \Omega\ \angle -90°} = 5\ \text{A}\ \angle 90°$$

and

$$\mathbf{I}'' = \frac{\mathbf{I}_{s_2}}{2} = 2.5\ \text{A}\ \angle 90°$$



The resultant current through the 4-Ω reactance X_{L2} (Fig. 5) is

Fig. 5

$$\mathbf{I} = \mathbf{I}' - \mathbf{I}''$$
$$= 3.75\ \text{A}\ \angle -90° - 2.50\ \text{A}\ \angle 90° = -j\,3.75\ \text{A} - j\,2.50\ \text{A}$$
$$= -j\,6.25\ \text{A}$$
$$\mathbf{I} = \mathbf{6.25\ A}\ \angle \mathbf{-90°}$$

2. THE´VENIN'S THEOREM

Thévenin's theorem, ***any two-terminal linear ac network can be replaced with an equivalent circuit consisting of a voltage source and an impedance in series, as shown in Fig. 1.***

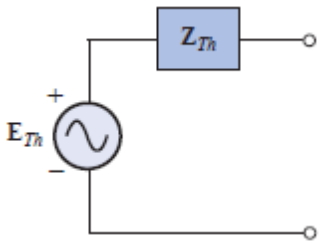
Steps of Solution:

1. Remove that portion of the network across which the Thévenin equivalent circuit is to be found.

2. Mark (○) the terminals of the remaining two-terminal network.

3. Calculate $\mathbf{Z}_{Th}$ by first setting all voltage and current sources to zero (short circuit and open circuit, respectively) and then finding the resulting impedance between the two marked terminals.

4. Calculate $\mathbf{E}_{Th}$ by first replacing the voltage and current sources



and then finding the open-circuit voltage between the marked terminals.
5. Draw the Thévenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the Thévenin equivalent circuit.

Fig.1

EXAMPLE 1 Find the Thévenin equivalent circuit for the network external to resistor *R* in Fig. 1.

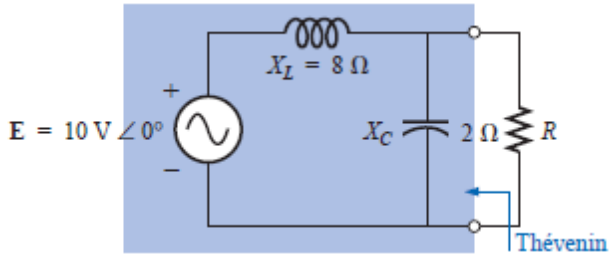


Fig.1

Solution

Step 1 and 2 (Fig.2)

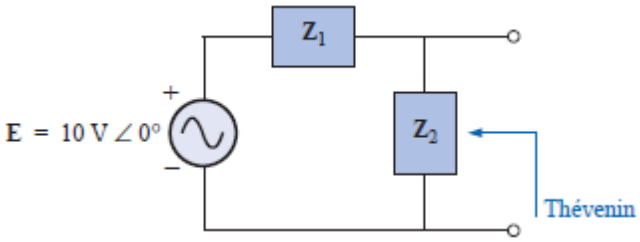


Fig.2

$\mathbf{Z}_1 = jX_L = j\,8\ \Omega \qquad \mathbf{Z}_2 = -jX_C = -j\,2\ \Omega$

Step 3 (Fig.3)

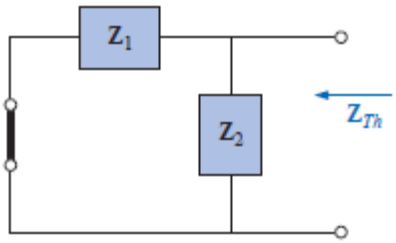
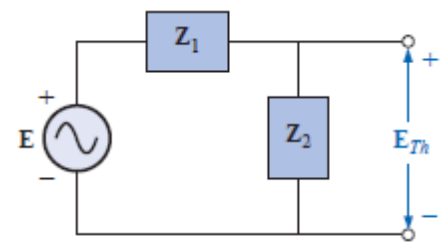


Fig.3

$$\mathbf{Z}_{Th} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(j\,8\ \Omega)(-j\,2\ \Omega)}{j\,8\ \Omega - j\,2\ \Omega} = \frac{-j^2 16\ \Omega}{j\,6} = \frac{16\ \Omega}{6\ \angle 90°}$$
$$= 2.67\ \Omega\ \angle -90°$$

Step 4 (Fig.4)



$$\mathbf{E}_{Th} = \frac{\mathbf{Z}_2\mathbf{E}}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad \text{(voltage divider rule)}$$

$$= \frac{(-j\,2\ \Omega)(10\ \text{V})}{j\,8\ \Omega - j\,2\ \Omega} = \frac{-j\,20\ \text{V}}{j\,6} = \mathbf{3.33\ V\ \angle -180°}$$

Fig.4

Step 5: The Thévenin equivalent circuit is shown in Fig. 5.

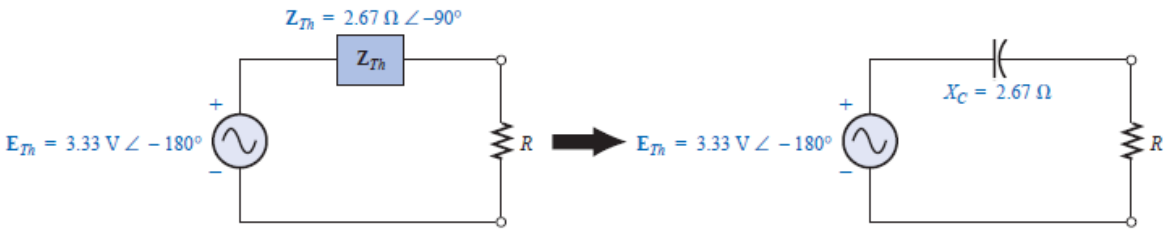


Fig.5

NORTON'S THEOREM

The method described for Thévenin's theorem will be altered to permit its use with **Norton's theorem.** Since the Thévenin and Norton impedances are the same for a particular network, certain portions of the discussion will be quite similar to those encountered in the previous section. Norton's theorem allows us to replace any two-terminal linear bilateral ac network with an equivalent circuit consisting of a current source and an impedance, as in Fig. 6.

The Norton equivalent circuit, like the Thévenin equivalent circuit, is applicable at only one frequency since the reactances are frequency dependent.

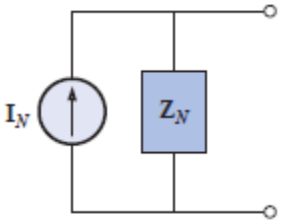


Fig.1*The Norton equivalent circuit for ac networks.*

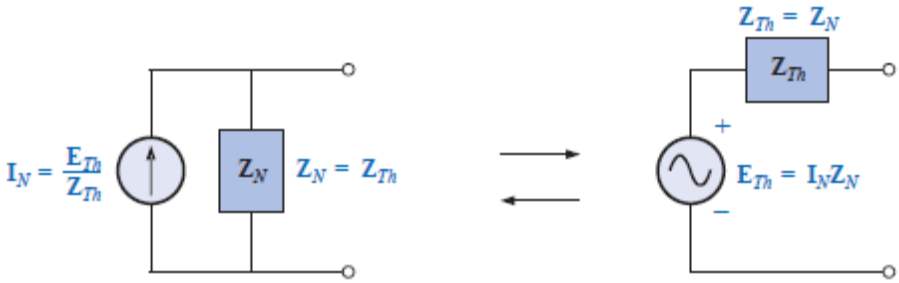
Steps of solution

1. Remove that portion of the network across which the Norton equivalent circuit is to be found.

2. Mark (○) the terminals of the remaining two-terminal network.
3. Calculate Z_N by first setting all voltage and current sources to zero (short circuit and open circuit, respectively) and then finding the resulting impedance between the two marked terminals.
4. Calculate I_N by first replacing the voltage and current sources and then finding the short-circuit current between the marked terminals.
5. Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the Norton equivalent circuit.

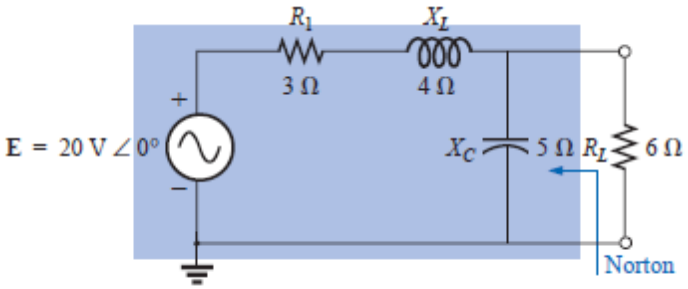
- The Norton and Thévenin equivalent circuits can be found from each other by using the source transformation shown in Fig. 7. The source transformation is applicable for any Thévenin or Norton equivalent circuit determined from a network.

Fig.7



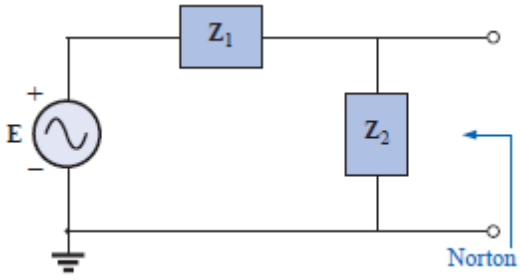
EXAMPLE 18.14 Determine the Norton equivalent circuit for the network external to the 6-Ω resistor of Fig. 8.

Fig.8



Solution :

Step 1 and 2 : (Fig.9)



$$\mathbf{Z}_1 = R_1 + j\,X_L = 3\ \Omega + j\,4\ \Omega = 5\ \Omega \angle 53.13°$$
$$\mathbf{Z}_2 = -j\,X_C = -j\,5\ \Omega$$

Fig.9

Step 3 : (Fig. 10)

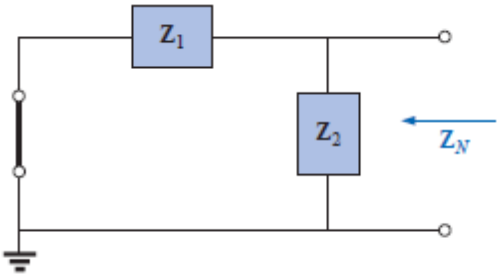


Fig.10

$$\mathbf{Z}_N = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(5\ \Omega\angle 53.13°)(5\ \Omega\angle -90°)}{3\ \Omega + j\,4\ \Omega - j\,5\ \Omega} = \frac{25\ \Omega\angle -36.87°}{3 - j\,1}$$

$$= \frac{25\ \Omega\angle -36.87°}{3.16\angle -18.43°} = 7.91\ \Omega\angle -18.44° = \mathbf{7.50\ \Omega - j\,2.50\ \Omega}$$

Step 4: (Fig.11)

$$\mathbf{I}_N = \mathbf{I}_1 = \frac{\mathbf{E}}{\mathbf{Z}_1} = \frac{20\ \text{V}\angle 0°}{5\ \Omega\angle 53.13°} = \mathbf{4\ A\angle -53.13°}$$

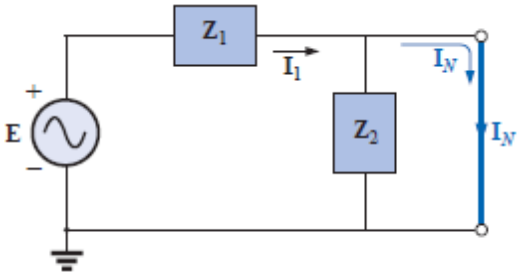


Fig.11

Step 5: The Norton equivalent circuit is shown in Fig. 12.

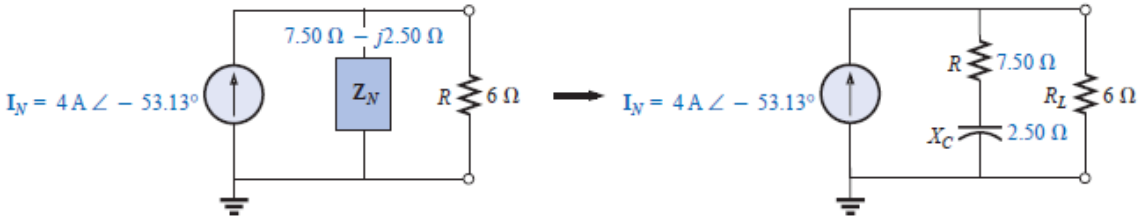


Fig.12

MAXIMUM POWER TRANSFER THEOREM

When applied to ac circuits, the **maximum power transfer theorem** states that :
maximum power will be delivered to a load when the load impedance is the conjugate of the Thévenin impedance across its terminals.
That is, for Fig. 13, for maximum power transfer to the load,

$$Z_L = Z_{Th} \quad \text{and} \quad \theta_L = -\theta_{Th_Z}$$

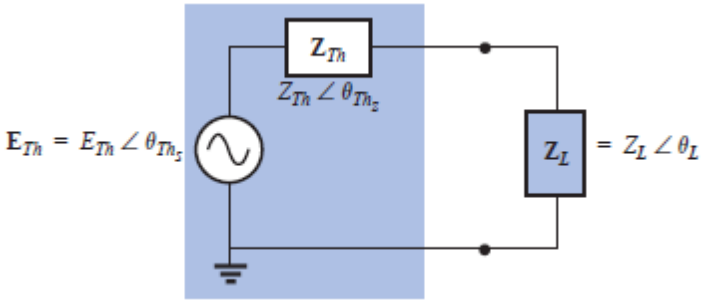


Fig.13

Or in rectangular form

$$R_L = R_{Th} \quad \text{and} \quad \pm j X_{\text{load}} = \mp j X_{Th}$$

The conditions just mentioned will make the total impedance of the circuit appear purely resistive, as indicated in Fig. 14:

$$\mathbf{Z}_T = (R \pm j X) + (R \mp j X)$$

$$\mathbf{Z}_T = 2R$$

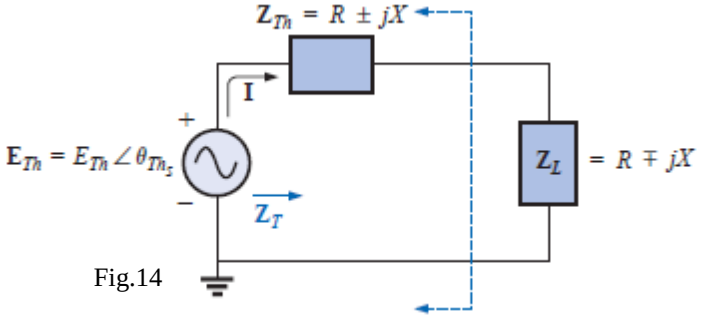


Fig.14

Since the circuit is purely resistive, the power factor of the circuit under maximum power conditions is 1; that is,

$$F_p = 1$$ (maximum power transfer)

The magnitude of the current **I** of Fig. 14 is

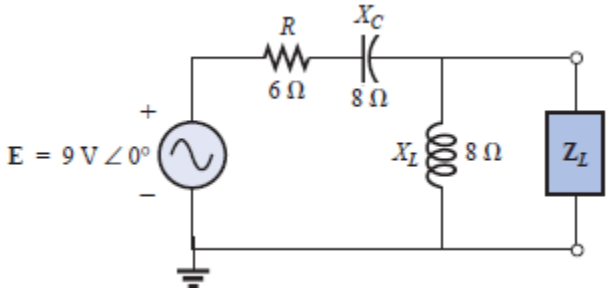
$$I = \frac{E_{Th}}{Z_T} = \frac{E_{Th}}{2R}$$

The maximum power to the load is $P_{\max} = I^2 R = \left(\frac{E_{Th}}{2R}\right)^2 R$

$$P_{\max} = \frac{E_{Th}^2}{4R}$$

EXAMPLE 18.19 Find the load impedance in Fig. 15 for maximum power to the load, and find the maximum power.

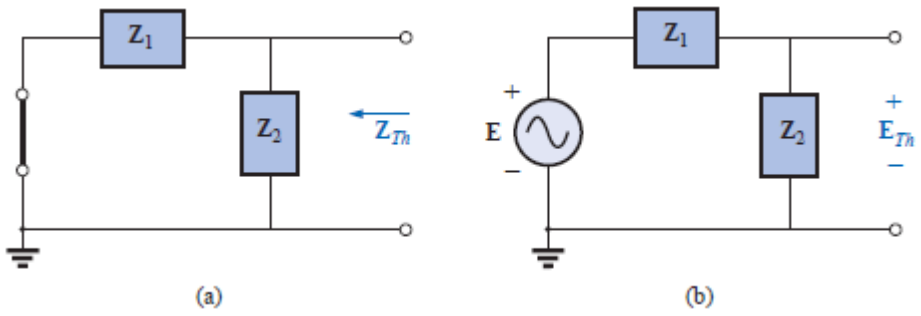
Fig.15



Solution: Determine $\mathbf{Z}_{Th}$ [Fig. 16(a)]: $\mathbf{Z}_1 = R - jX_C = 6\ \Omega - j\,8\ \Omega = 10\ \Omega \angle -53.13°$

$\mathbf{Z}_2 = +jX_L = j\,8\ \Omega$

Fig.16



$$\mathbf{Z}_{Th} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(10\ \Omega \angle -53.13°)(8\ \Omega \angle 90°)}{6\ \Omega - j\,8\ \Omega + j\,8\ \Omega} = \frac{80\ \Omega \angle 36.87°}{6 \angle 0°}$$

$$= 13.33\ \Omega \angle 36.87° = 10.66\ \Omega + j\,8\ \Omega$$

and $\mathbf{Z}_L = 13.3\ \Omega \angle -36.87° = \mathbf{10.66\ \Omega - j\,8\ \Omega}$

To find the maximum power, we must first find $\mathbf{E}_{Th}$ [Fig. 16(b)], as follows:

$$\mathbf{E}_{Th} = \frac{\mathbf{Z}_2\mathbf{E}}{\mathbf{Z}_2 + \mathbf{Z}_1} \quad \text{(voltage divider rule)}$$

$$= \frac{(8\ \Omega \angle 90°)(9\ \text{V} \angle 0°)}{j\,8\ \Omega + 6\ \Omega - j\,8\ \Omega} = \frac{72\ \text{V} \angle 90°}{6 \angle 0°} = 12\ \text{V} \angle 90°$$

Then $P_{\max} = \frac{E_{Th}^2}{4R} = \frac{(12\ \text{V})^2}{4(10.66\ \Omega)} = \frac{144}{42.64} = \mathbf{3.38\ W}$

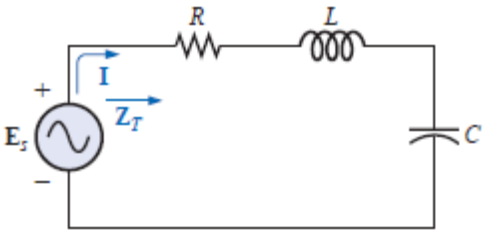
Resonance

1.SERIES RESONANCE

1.2 SERIES RESONANT CIRCUIT

This section will introduce the very important *resonant* (or *tuned*) *circuit*, which is fundamental to the operation of a wide variety of electrical and electronic systems in use today. The resonant circuit is a combination of *R*,
L, and *C* elements. The basic configuration for the series resonant circuit appears in Fig. 1
.

The total impedance of this network at any frequency is determined by



$$\mathbf{Z}_T = R + j X_L - j X_C = R + j (X_L - X_C) \quad (1)$$

The resonant conditions described in the introduction will occur when

$$X_L = X_C \quad (2)$$

Fig.1

removing the reactive component from the total impedance equation. The total impedance at resonance is then simply

$$\mathbf{Z}_{T_s} = R \quad (3)$$

representing the minimum value of $\mathbf{Z}_T$ at any frequency. The subscript *s* will be employed to indicate series resonant conditions. The resonant frequency can be determined in terms of the inductance and capacitance by examining the defining equation for resonance [Eq. (2)]:

$$X_L = X_C$$

Substituting yields $\omega L = \dfrac{1}{\omega C}$ and $\omega^2 = \dfrac{1}{LC}$

and $$\omega_s = \frac{1}{\sqrt{LC}} \quad (4)$$

or $$f_s = \frac{1}{2\pi\sqrt{LC}} \quad (5)$$

f = hertz (Hz)
L = henries (H)
C = farads (F)

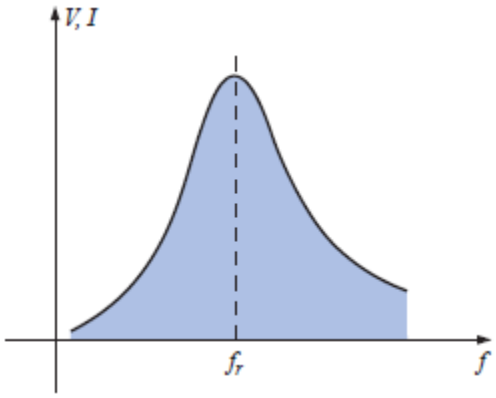
The current through the circuit at resonance is

$$\mathbf{I} = \frac{E \angle 0°}{R \angle 0°} = \frac{E}{R} \angle 0°$$

which you will note is the maximum current for the circuit of Fig. 1for an applied voltage **E** since **Z***T* is a minimum value.

The frequency response characteristic of the circuit is shown in Fig. 2. Note in the figure that the response is a maximum for the frequency *fr* (resonant frequency = f_s) .

Fig.2



1.2 THE QUALITY FACTOR (*Q*)

The **quality factor** *Q* of a series resonant circuit is defined as the ratio of the reactive power of either the inductor or the capacitor to the average power of the resistor at resonance; that is,

$$Q_s = \frac{\text{reactive power}}{\text{average power}}$$

Hence at resonant Q factor is given l

$$Q_s = \frac{I^2 X_L}{I^2 R}$$

$$Q_s = \frac{X_L}{R} = \frac{\omega_s L}{R}$$

or

$$Q_s = \frac{\omega_s L}{R} = \frac{2\pi f_s L}{R} = \frac{2\pi}{R}\left(\frac{1}{2\pi\sqrt{LC}}\right)L$$

$$= \frac{L}{R}\left(\frac{1}{\sqrt{LC}}\right) = \left(\frac{\sqrt{L}}{\sqrt{L}}\right)\frac{L}{R\sqrt{LC}}$$

and

$$Q_s = \frac{1}{R}\sqrt{\frac{L}{C}}$$

EXAMPLE 1

a. For the series resonant circuit of Fig. 2, find **I**, **V***R*, **V***L*, and **V***C* at resonance.
b. What is the *Qs* of the circuit?

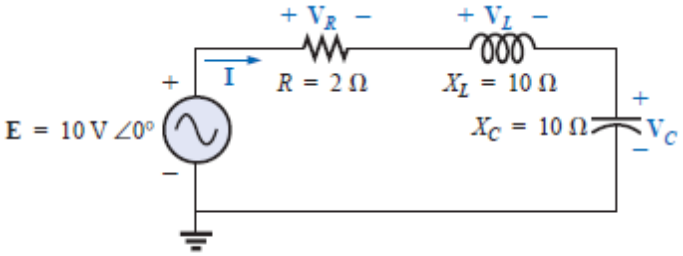


Fig.2

Solution

a. $\mathbf{Z}_{T_s} = R = 2\ \Omega$

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_{T_s}} = \frac{10\ \text{V}\angle 0^\circ}{2\ \Omega\angle 0^\circ} = \mathbf{5\ A\angle 0^\circ}$$

$$\mathbf{V}_R = \mathbf{E} = 10\ \text{V}\angle 0^\circ$$

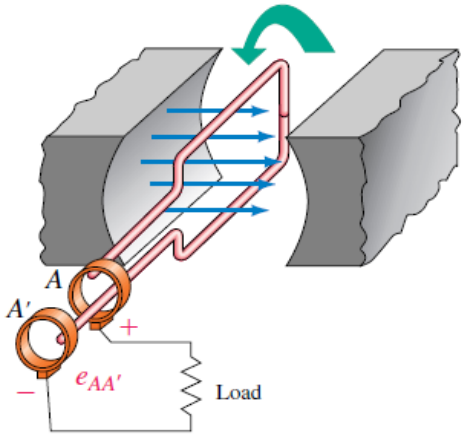
$$\mathbf{V}_L = (I\angle 0^\circ)(X_L\angle 90^\circ) = (5\ \text{A}\angle 0^\circ)(10\ \Omega\angle 90^\circ) = \mathbf{50\ V\angle 90^\circ}$$

$$\mathbf{V}_C = (I\angle 0^\circ)(X_C\angle -90^\circ) = (5\ \text{A}\angle 0^\circ)(10\ \Omega\angle -90^\circ) = \mathbf{50\ V\angle -90^\circ}$$

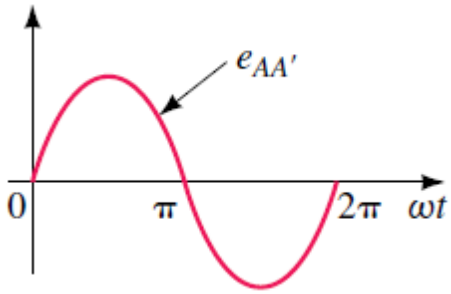
b. $Q_s = \frac{X_L}{R} = \frac{10\ \Omega}{2\ \Omega} = \mathbf{5}$

Three phase systems

- The voltage induced by a single coil when rotated in a uniform magnetic field is shown in Figure 1 and is known as a **single-phase voltage**.
- **three-phase supply** is generated when three coils are placed 120° apart and the whole rotated in a uniform magnetic field as shown in Figure 2(a). The result is three independent supplies of equal voltages which are each displaced by 120° from each other as shown in Figure 2(b).

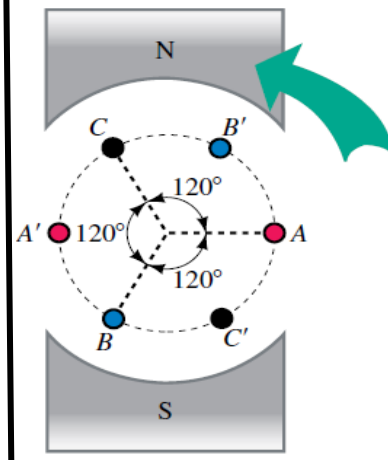


(a) Basic single –phase generator

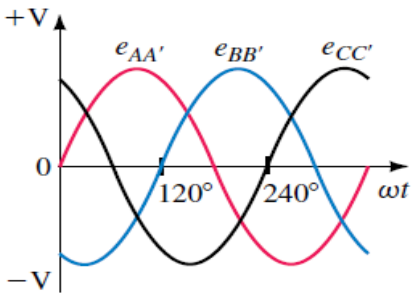


(b) Voltage waveform

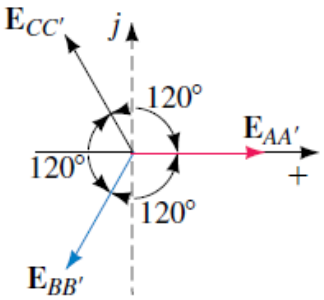
Fig.1



(a) Basic 3-phase generator



(b) Voltage waveforms



(c) Phasors

Fig.2

If the three-phase windings shown in Fig.2 are kept independent then six wires are needed to connect a supply source to a load (Fig.3) . To reduce the number of wires it is usual to interconnect the three phases. There are two ways in which this can be done, these being:
(a) a **star, or Y, connection**, and (b) a **delta**, or **mesh, connection**.

Points A′ B′ and C′ may be replaced by N

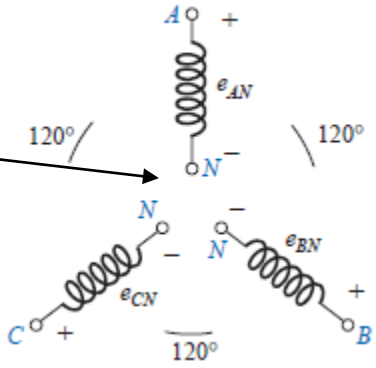


Fig.3

THE Y-CONNECTED SOURCE (GENERATOR)

If the three terminals denoted *N* of Fig. 3 are connected together, the source (generator) is referred to as a **Y-connected three-phase source (generator). (Fg.4).**

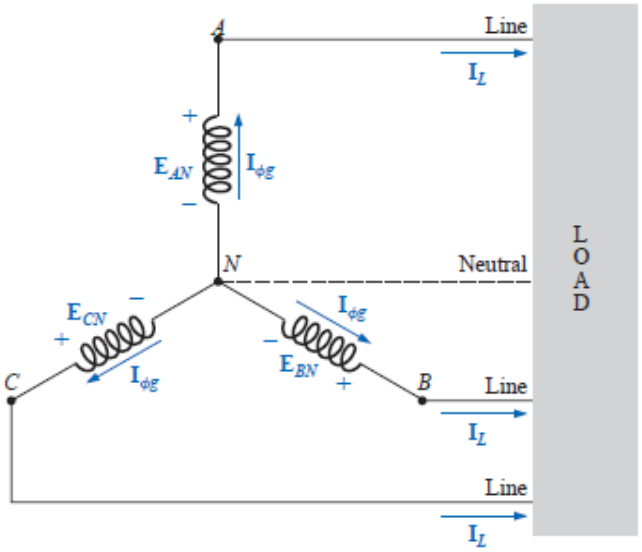


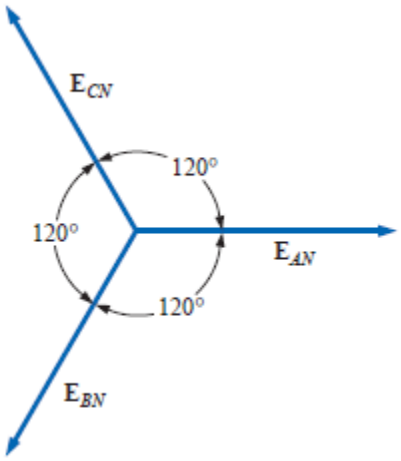
Fig.4

The point at which all the terminals are connected is called the ***neutral point.*** If a conductor is not attached from this point to the load, the system is called a ***Y-connected, three-phase, three-wire system***. If the neutral is connected, the system is a ***Y-connected, three-phase, four-wire system***.

The phase voltages and line voltages

The three generated voltages in each phase are:

$$e_{AN} = E_{m(AN)} \sin \omega t$$
$$e_{BN} = E_{m(BN)} \sin(\omega t - 120°)$$
$$e_{CN} = E_{m(CN)} \sin(\omega t - 240°) = E_{m(CN)} \sin(\omega t + 120°)$$



Where e_{AN} , e_{BN} and e_{CN} are the phase voltages

The phasor diagram of the induced voltages is shown in Fig. 5,where the rms (effective) value of each is determined by

Fig.5

$$E_{AN} = 0.707E_{m(AN)}$$
$$E_{BN} = 0.707E_{m(BN)}$$
$$E_{CN} = 0.707E_{m(CN)}$$

and

$$\mathbf{E}_{AN} = E_{AN} \angle 0°$$
$$\mathbf{E}_{BN} = E_{BN} \angle -120°$$
$$\mathbf{E}_{CN} = E_{CN} \angle +120°$$

By rearranging the phasors as shown in Fig. 6 and applying a law of vectors which states that *the vector sum of any number of vectors drawn such that the "head" of one is connected to the "tail" of the next, and that the head of the last vector is connected to the tail of the first is zero*, we can conclude that the phasor sum of the phase voltages in a three-phase system is zero. That is,

$$\mathbf{E}_{AN} + \mathbf{E}_{BN} + \mathbf{E}_{CN} = 0$$

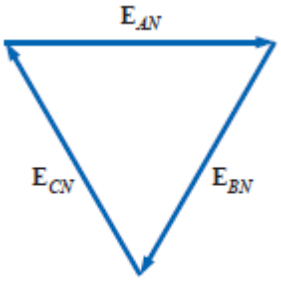


Fig.6

The voltage from one line to another is called a **line voltage.** On the phasor diagram (Fig. 7) it is the phasor drawn from the end of one phase to another in the counterclockwise direction.

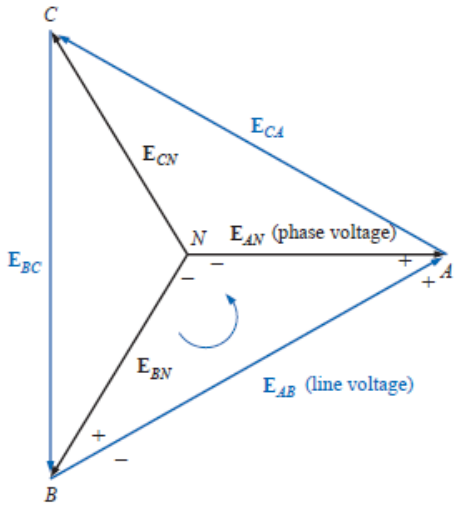


Fig.7

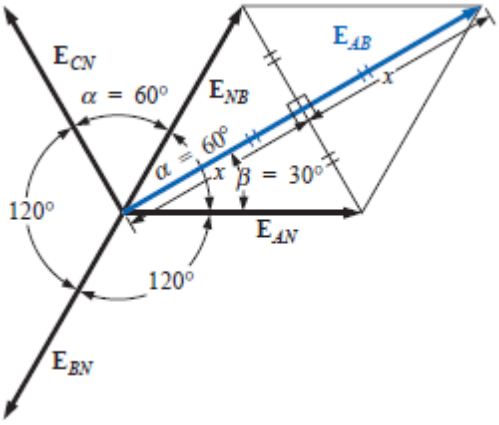


Fig.8

Applying Kirchhoff's voltage law around the indicated loop of Fig. 7, we obtain

$$\mathbf{E}_{AB} - \mathbf{E}_{AN} + \mathbf{E}_{BN} = 0$$

Or $$\mathbf{E}_{AB} = \mathbf{E}_{AN} - \mathbf{E}_{BN} = \mathbf{E}_{AN} + \mathbf{E}_{NB}$$

The phasor diagram is redrawn to find $\mathbf{E}_{AB}$ as shown in Fig. 8. Since each phase voltage, when reversed ($\mathbf{E}_{NB}$), will bisect the other two, $\alpha = 60°$. The angle β is 30° since a line drawn from opposite ends of a rhombus will divide in half both the angle of origin and the opposite
angle. Lines drawn between opposite corners of a rhombus will also bisect each other at right angles.
The length x is

$$x = E_{AN}\cos 30° = \frac{\sqrt{3}}{2}E_{AN}$$

Noting from the phas$E_{AB} = 2x = (2)\frac{\sqrt{3}}{2}E_{AN} = \sqrt{3}E_{AN}$ 30°, the result is

$$\mathbf{E}_{AB} = E_{AB}\angle 30° = \sqrt{3}E_{AN}\angle 30°$$
$$\mathbf{E}_{CA} = \sqrt{3}E_{CN}\angle 150°$$
$$\mathbf{E}_{BC} = \sqrt{3}E_{BN}\angle 270°$$

In words, the magnitude of the line voltage of a Y-connected generator is $\sqrt{3}$ times the phase voltage:

$$E_L = \sqrt{3}E_\phi$$

with the phase angle between any line voltage and the nearest phase voltage at 30°.
In sinusoidal notation,

$$e_{AB} = \sqrt{2}E_{AB}\sin(\omega t + 30°)$$
$$e_{CA} = \sqrt{2}E_{CA}\sin(\omega t + 150°)$$
$$e_{BC} = \sqrt{2}E_{BC}\sin(\omega t + 270°)$$

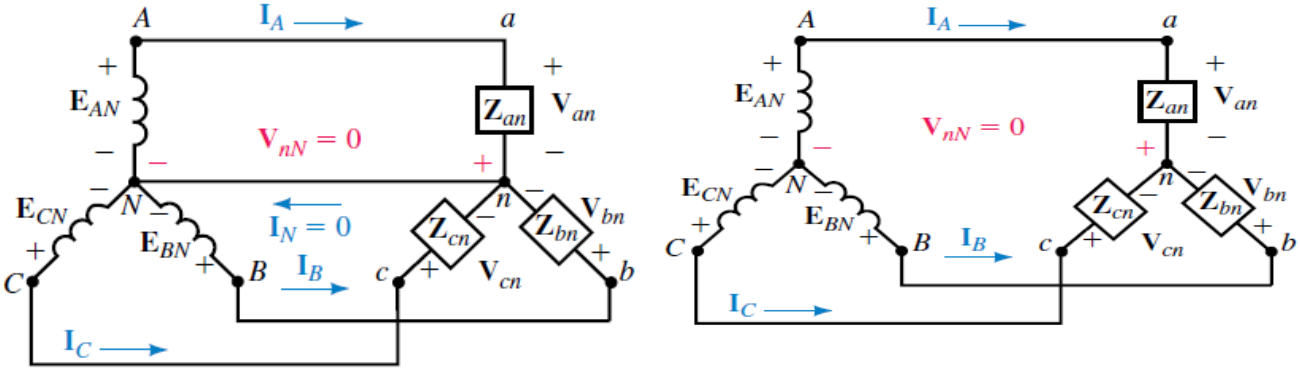
Line Current and Phase Current

The three conductors connected from *A*, *B*, and *C* to the load are called *lines*. For the Y-connected system, it should be obvious from Fig. 4 that the **line current** equals the **phase current** for each phase; that is,

$$\mathbf{I}_L = \mathbf{I}_{\phi g}$$

where ϕ is used to denote a phase quantity and *g* is a generator parameter.

- Y-Y connection may take two forms : 3-wire or 4-Wire system as shown in Fig.9



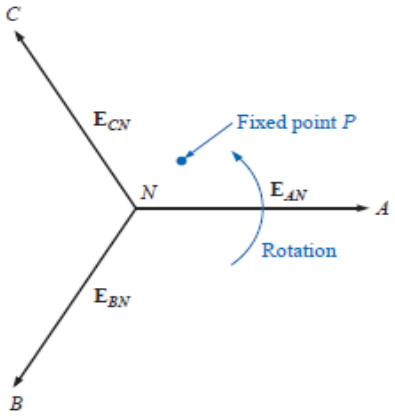
(a) 4-Wire Y-Y connection (b) 3-Wire Y-Y connection

Fig.9

Phase Sequence

Phase sequence refers to the order in which three-phase voltages are generated. Consider again Figure 2. As the rotor turns in the counterclockwise direction, voltages are generated in the sequence e_{AN}, e_{BN}, and e_{CN} and the system is said to have an **ABC phase sequence.** On the other hand, if the direction of rotation were reversed, the sequence would be *ACB* . Sequence *ABC* is called the **positive phase sequence** and is the sequence generated in practice as shown in Fig.10. It is therefore the only sequence considered in this section .

Fig.10 Positive sequence *ABC*



EXAMPLE 1 The phase sequence of the Y-connected generator in Fig. 11 is *ABC*.
a. Find the phase angles θ_2 and θ_3.

b. Find the magnitude of the line voltages.
c. Find the line currents.
d. Verify that, since the load is balanced, $\mathbf{I}_N = 0$.

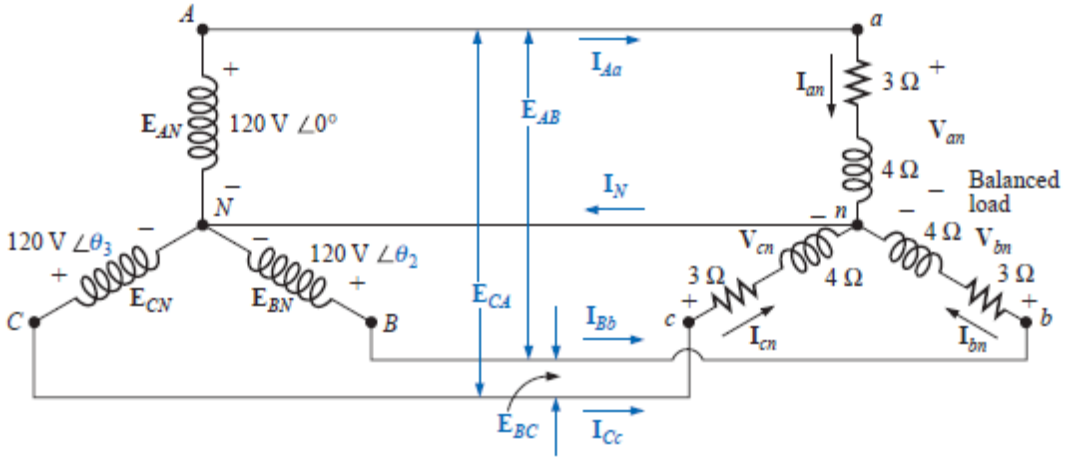


Fig.11

Solutions:

a. For an *ABC* phase sequence,

$\theta_2 = \mathbf{-120°}$ and $\theta_3 = \mathbf{+120°}$

b. $E_L = \sqrt{3}E_\phi = (1.73)(120\text{ V}) = 208\text{ V}$. Therefore,

$$E_{AB} = E_{BC} = E_{CA} = \mathbf{208\text{ V}}$$

c. $\mathbf{V}_\phi = \mathbf{E}_\phi$. Therefore,

$$\mathbf{V}_{an} = \mathbf{E}_{AN} \qquad \mathbf{V}_{bn} = \mathbf{E}_{BN} \qquad \mathbf{V}_{cn} = \mathbf{E}_{CN}$$

$$\mathbf{I}_{\phi L} = \mathbf{I}_{an} = \frac{\mathbf{V}_{an}}{\mathbf{Z}_{an}} = \frac{120\text{ V}\angle 0°}{3\ \Omega + j\,4\ \Omega} = \frac{120\text{ V}\angle 0°}{5\ \Omega\angle 53.13°}$$

$$= 24\text{ A}\angle -53.13°$$

$$\mathbf{I}_{bn} = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_{bn}} = \frac{120\text{ V}\angle -120°}{5\ \Omega\angle 53.13°} = 24\text{ A}\angle -173.13°$$

$$\mathbf{I}_{cn} = \frac{\mathbf{V}_{cn}}{\mathbf{Z}_{cn}} = \frac{120\text{ V}\angle +120°}{5\ \Omega\angle 53.13°} = 24\text{ A}\angle 66.87°$$

and, since $\mathbf{I}_L = \mathbf{I}_{\phi L}$,

$$\mathbf{I}_{Aa} = \mathbf{I}_{an} = \mathbf{24\text{ A}\angle -53.13°}$$
$$\mathbf{I}_{Bb} = \mathbf{I}_{bn} = \mathbf{24\text{ A}\angle -173.13°}$$
$$\mathbf{I}_{Cc} = \mathbf{I}_{cn} = \mathbf{24\text{ A}\angle 66.87°}$$

d. Applying Kirchhoff's current law, we have

$$\mathbf{I}_N = \mathbf{I}_{Aa} + \mathbf{I}_{Bb} + \mathbf{I}_{Cc}$$

In rectangular form,

$$\begin{aligned}
\mathbf{I}_{Aa} &= 24\text{ A} \angle -53.13^\circ &&= 14.40\text{ A} - j\,19.20\text{ A} \\
\mathbf{I}_{Bb} &= 24\text{ A} \angle -173.13^\circ &&= -22.83\text{ A} - j\,2.87\text{ A} \\
\mathbf{I}_{Cc} &= 24\text{ A} \angle 66.87^\circ &&= 9.43\text{ A} + j\,22.07\text{ A} \\
\Sigma(\mathbf{I}_{Aa} + \mathbf{I}_{Bb} + \mathbf{I}_{Cc}) &&&= 0 + j\,0
\end{aligned}$$

and $\mathbf{I}_N$ is in fact equal to **zero**, as required for a balanced load.

THE Y-Δ SYSTEM

There is no neutral connection for the Y-Δ system of Fig. 12. Any variation in the impedance of a phase that produces an unbalanced system will simply vary the line and phase currents of the system.

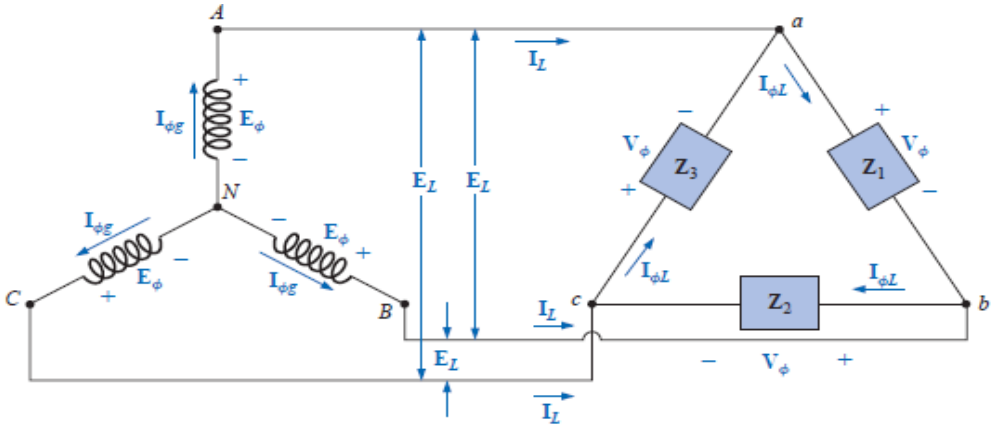


Fig.12

For a balanced load,

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3$$

The voltage across each phase of the load is equal to the line voltage of the generator for a balanced or an unbalanced load:

$$\mathbf{V}_\phi = \mathbf{E}_L$$

The relationship between the line currents and phase currents of a balanced Δ load is:

$$I_L = \sqrt{3} I_\phi$$

- For a balanced load, the line currents will be equal in magnitude, as will the phase currents.

EXAMPLE 2 For the three-phase system of Fig. 12:
a. Find the phase angles θ_2 and θ_3.
b. Find the current in each phase of the load.
c. Find the magnitude of the line currents.

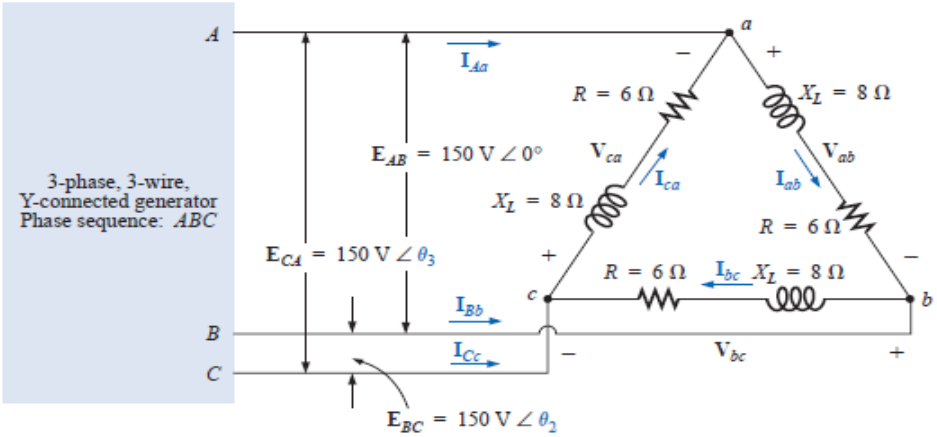


Fig.12

Solutions:

a. For an *ABC* sequence, $\theta_2 = -120°$ and $\theta_3 = +120°$

b. $\mathbf{V}_\phi = \mathbf{E}_L$. Therefore,

$$\mathbf{V}_{ab} = \mathbf{E}_{AB} \qquad \mathbf{V}_{ca} = \mathbf{E}_{CA} \qquad \mathbf{V}_{bc} = \mathbf{E}_{BC}$$

The phase currents are

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{150\text{ V} \angle 0°}{6\ \Omega + j\,8\ \Omega} = \frac{150\text{ V} \angle 0°}{10\ \Omega \angle 53.13°} = \mathbf{15\text{ A} \angle -53.13°}$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc}} = \frac{150\text{ V} \angle -120°}{10\ \Omega \angle 53.13°} = \mathbf{15\text{ A} \angle -173.13°}$$

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca}} = \frac{150\text{ V} \angle +120°}{10\ \Omega \angle 53.13°} = \mathbf{15\text{ A} \angle 66.87°}$$

c. $I_L = \sqrt{3} I_\phi = (1.73)(15\text{ A}) = 25.95\text{ A}$. Therefore,

$$I_{Aa} = I_{Bb} = I_{Cc} = \mathbf{25.95\text{ A}}$$

POWER CALCULATION IN 3- PHASE SYSTEM

- **Y-Connected Balanced Load**

Consider the balanced Y-cnnected load shown in Fig.13.

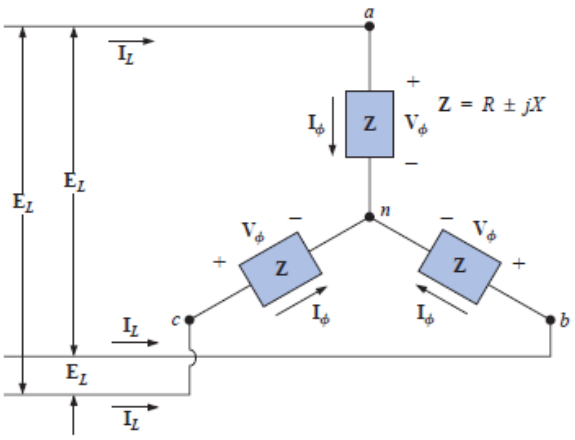


Fig.13

Average Power The average power delivered to each phase can be determined by:

$$P_\phi = V_\phi I_\phi \cos\theta_{I_\phi}^{V_\phi} = I_\phi^2 R_\phi = \frac{V_R^2}{R_\phi}$$ Watts

Where $\theta_{I_\phi}^{V_\phi}$ is the phase angle between the voltage and current.

The total power to the balanced load is

$$P_T = 3P_\phi \quad \text{(W)}$$

or, since $V_\phi = \frac{E_L}{\sqrt{3}}$ and $I_\phi = I_L$

then $$P_T = 3\frac{E_L}{\sqrt{3}} I_L \cos\theta_{I_\phi}^{V_\phi}$$

But $$\left(\frac{3}{\sqrt{3}}\right)(1) = \left(\frac{3}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

Therefore,

$$P_T = \sqrt{3} E_L I_L \cos\theta_{I_\phi}^{V_\phi} = 3I_L^2 R_\phi \quad \text{(W)}$$

Reactive Power The reactive power of each phase (in volt-amperes reactive) is

$$Q_\phi = V_\phi I_\phi \sin\theta_{I_\phi}^{V_\phi} = I_\phi^2 X_\phi = \frac{V_X^2}{X_\phi}$$ (VAR)

The total reactive power of the load is

$$\boxed{Q_T = 3Q_\phi} \quad \text{(VAR)}$$

or, proceeding in the same manner as above, we have

$$\boxed{Q_T = \sqrt{3}E_L I_L \sin\theta_{I_\phi}^{V_\phi} = 3I_L^2 X_\phi} \quad \text{(VAR)}$$

Apparent Power The apparent power of each phase is

$$\boxed{S_\phi = V_\phi I_\phi} \quad \text{(VA)}$$

The total apparent power of the load is

$$\boxed{S_T = 3S_\phi} \quad \text{(VA)}$$

or, as before,

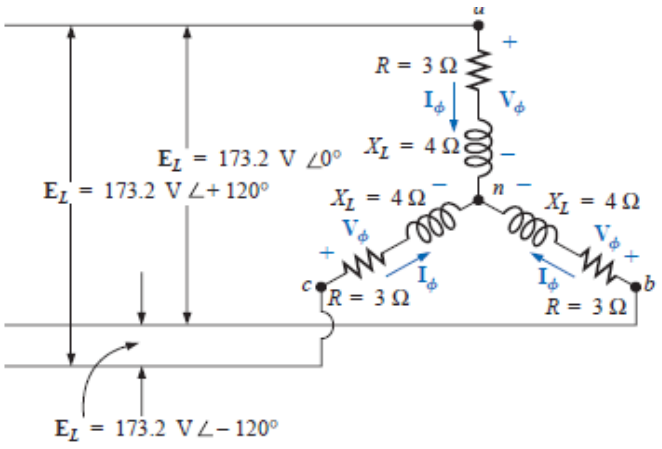
$$\boxed{S_T = \sqrt{3}E_L I_L} \quad \text{(VA)}$$

Power Factor The power factor of the system is given by

$$\boxed{F_p = \frac{P_T}{S_T} = \cos\theta_{I_\phi}^{V_\phi} \quad \text{(leading or lagging)}}$$

EXAMPLE 3 For the Y-connected load of Fig. 14:

Fig.14



a. Find the average power to each phase and the total load.
b. Determine the reactive power to each phase and the total reactive power.
c. Find the apparent power to each phase and the total apparent power.
d. Find the power factor of the load.

Solutions:

a. The *average power* is

$$P_\phi = V_\phi I_\phi \cos\theta_{I_\phi}^{V_\phi} = (100\text{ V})(20\text{ A})\cos 53.13° = (2000)(0.6)$$
$$= \mathbf{1200\text{ W}}$$
$$P_\phi = I_\phi^2 R_\phi = (20\text{ A})^2(3\ \Omega) = (400)(3) = \mathbf{1200\text{ W}}$$

or

$$P_\phi = \frac{V_R^2}{R_\phi} = \frac{(60\text{ V})^2}{3\ \Omega} = \frac{3600}{3} = \mathbf{1200\text{ W}}$$
$$P_T = 3P_\phi = (3)(1200\text{ W}) = \mathbf{3600\text{ W}}$$

or

$$P_T = \sqrt{3}E_L I_L \cos\theta_{I_\phi}^{V_\phi} = (1.732)(173.2\text{ V})(20\text{ A})(0.6) = \mathbf{3600\text{ W}}$$

b. The *reactive power* is

$$Q_\phi = V_\phi I_\phi \sin\theta_{I_\phi}^{V_\phi} = (100\text{ V})(20\text{ A})\sin 53.13° = (2000)(0.8)$$
$$= \mathbf{1600\text{ VAR}}$$

or $\quad Q_\phi = I_\phi^2 X_\phi = (20\text{ A})^2(4\ \Omega) = (400)(4) = \mathbf{1600\text{ VAR}}$
$$Q_T = 3Q_\phi = (3)(1600\text{ VAR}) = \mathbf{4800\text{ VAR}}$$

or

$$Q_T = \sqrt{3}E_L I_L \sin\theta_{I_\phi}^{V_\phi} = (1.732)(173.2\text{ V})(20\text{ A})(0.8) = \mathbf{4800\text{ VAR}}$$

c. The *apparent power* is

$$S_\phi = V_\phi I_\phi = (100\text{ V})(20\text{ A}) = \mathbf{2000\text{ VA}}$$
$$S_T = 3S_\phi = (3)(2000\text{ VA}) = \mathbf{6000\text{ VA}}$$

or $$S_T = \sqrt{3}E_L I_L = (1.732)(173.2\text{ V})(20\text{ A}) = \mathbf{6000\text{ VA}}$$

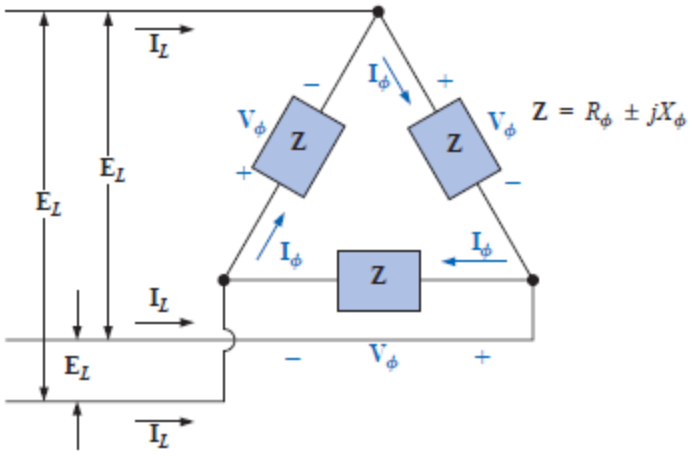
d. The *power factor* is

$$F_p = \frac{P_T}{S_T} = \frac{3600\text{ W}}{6000\text{ VA}} = \mathbf{0.6\ lagging}$$

- **Δ-Connected Balanced Load**

Consider the Δ-Connected Balanced Load shown in Fig.15

Fig.15



Average Power

$$P_\phi = V_\phi I_\phi \cos\theta_{I_\phi}^{V_\phi} = I_\phi^2 R_\phi = \frac{V_R^2}{R_\phi} \quad \text{(W)}$$

$$P_T = 3P_\phi \quad \text{(W)}$$

Reactive Power

$$Q_\phi = V_\phi I_\phi \sin\theta_{I_\phi}^{V_\phi} = I_\phi^2 X_\phi = \frac{V_X^2}{X_\phi} \quad \text{(VAR)}$$

$$Q_T = 3Q_\phi \quad \text{(VAR)}$$

Apparent Power

$$S_\phi = V_\phi I_\phi \quad \text{(VA)}$$

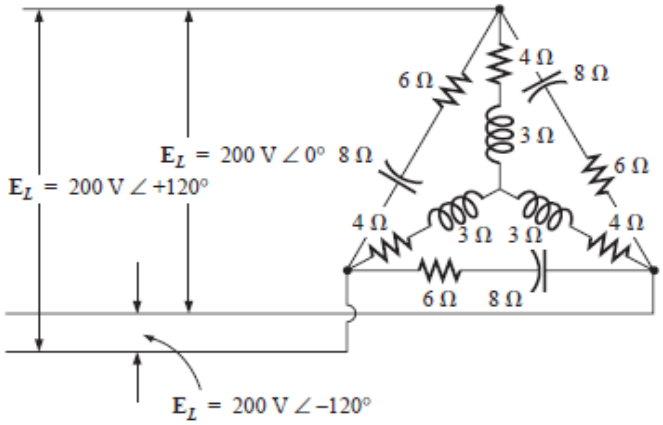
$$S_T = 3S_\phi = \sqrt{3}E_L I_L \quad \text{(VA)}$$

Power Factor

$$F_p = \frac{P_T}{S_T}$$

EXAMPLE 4 For the Δ -Y connected load of Fig. 16, find the total average, reactive, and apparent power. In addition, find the power factor of the load.

Fig.16



Solution: Consider the Δ and Y separately.
For the Δ:

$$\mathbf{Z}_\Delta = 6\ \Omega - j\,8\ \Omega = 10\ \Omega \angle -53.13°$$

$$I_\phi = \frac{E_L}{Z_\Delta} = \frac{200\ \text{V}}{10\ \Omega} = 20\ \text{A}$$

$$P_{T_\Delta} = 3I_\phi^2 R_\phi = (3)(20\ \text{A})^2(6\ \Omega) = \mathbf{7200\ W}$$

$$Q_{T_\Delta} = 3I_\phi^2 X_\phi = (3)(20\ \text{A})^2(8\ \Omega) = \mathbf{9600\ VAR}\ (C)$$

$$S_{T_\Delta} = 3V_\phi I_\phi = (3)(200\ \text{V})(20\ \text{A}) = \mathbf{12{,}000\ VA}$$

For the Y:

$$\mathbf{Z}_Y = 4\ \Omega + j\ 3\ \Omega = 5\ \Omega\ \angle 36.87°$$

$$I_\phi = \frac{E_L/\sqrt{3}}{Z_Y} = \frac{200\ \text{V}/\sqrt{3}}{5\ \Omega} = \frac{116\ \text{V}}{5\ \Omega} = 23.12\ \text{A}$$

$$P_{T_Y} = 3I_\phi^2 R_\phi = (3)(23.12\ \text{A})^2(4\ \Omega) = \mathbf{6414.41\ W}$$

$$Q_{T_Y} = 3I_\phi^2 X_\phi = (3)(23.12\ \text{A})^2(3\ \Omega) = \mathbf{4810.81\ VAR}\ (L)$$

$$S_{T_Y} = 3V_\phi I_\phi = (3)(116\ \text{V})(23.12\ \text{A}) = \mathbf{8045.76\ VA}$$

For the total load:

$$P_T = P_{T_\Delta} + P_{T_Y} = 7200\ \text{W} + 6414.41\ \text{W} = \mathbf{13{,}614.41\ W}$$

$$Q_T = Q_{T_\Delta} - Q_{T_Y} = 9600\ \text{VAR}\ (C) - 4810.81\ \text{VAR}\ (I)$$

$$= \mathbf{4789.19\ VAR}\ (C)$$

$$S_T = \sqrt{P_T^2 + Q_T^2} = \sqrt{(13{,}614.41\ \text{W})^2 + (4789.19\ \text{VAR})^2}$$

$$= \mathbf{14{,}432.2\ VA}$$

$$F_p = \frac{P_T}{S_T} = \frac{13{,}614.41\ \text{W}}{14{,}432.20\ \text{VA}} = \mathbf{0.943\ leading}$$

Measuring Power in Three-Phase Circuits

1-The Three-Wattmeter Method

Measuring power to a 4-wire Y load requires one wattmeter per phase (i.e., three wattmeters) as in Figure 1 . As indicated, wattmeter *W*1 is connected across voltage **V***an* and its current is **I***a*. Thus, its reading is

$$P_1 = V_{an} I_a \cos\theta_{an}$$

which is power to phase *an.* Similarly, *W*2 indicates power to phase *bn* and *W*3 to phase *cn.* Loads may be balanced or unbalanced. The total power is

$$P_T = P_1 + P_2 + P_3$$

If the load of Figure 23–28 could be guaranteed to always be balanced, only one wattmeter would be needed. *P*T would be 3 times its reading.

$$P_T = 3\,P_\phi$$

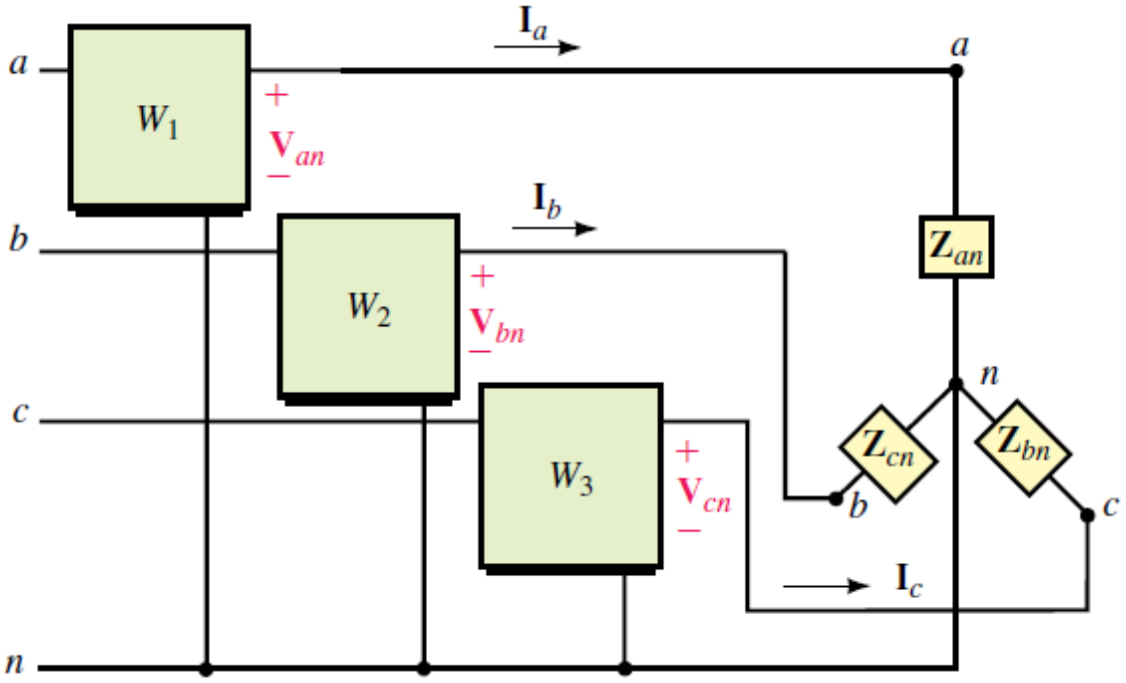


Fig.1 Three-wattmeter connection for a 4-wire load.

2-The Two-Wattmeter Method

While three wattmeters are required for a four-wire system, for a three-wire system, only two are needed. The connection is shown in Figure 2. Loads may be Y or Δ, balanced or unbalanced. The meters may be connected in any pair of lines with the voltage terminals connected to the third line. The total power is the algebraic sum of the meter readings.

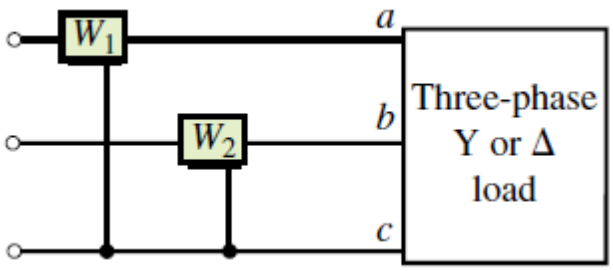
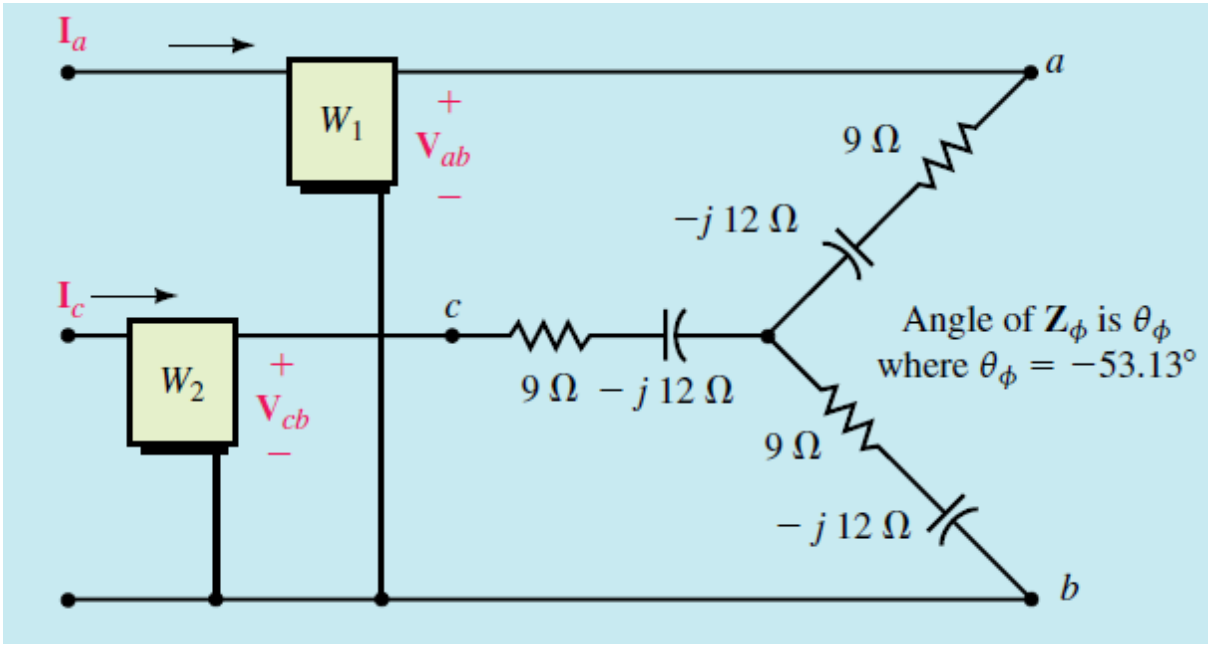


Fig.2 Two-wattmeter connection. Load may be balanced or unbalanced.

EXAMPLE 1 For Figure 3 , $\mathbf{V}_{an}$ = 120 V∠0°. Compute the readings of each meter, then sum to determine total power.



Solution $\mathbf{V}_{an}$=120V∠0°. Thus, $\mathbf{V}_{ab}$ =208V∠30° and $\mathbf{V}_{bc}$ =208V∠-90°.
$\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_{an}$ = 120 V∠0°/(9 - *j*12) Ω = 8 A∠53.13°. Thus, $\mathbf{I}_c$ = 8A∠173.13°.
First consider wattmeter 1, Figure 4. Note that *W*1 is connected to terminals *a-b;* thus it has voltage $\mathbf{V}_{ab}$ across it and current $\mathbf{I}_a$ through it. Its reading is therefore $P_1 = V_{ab} I_a \cos \theta_1$, where v_1 is the angle between $\mathbf{V}_{ab}$ and $\mathbf{I}_a$. $\mathbf{V}_{ab}$ has an angle of 30° and $\mathbf{I}_a$ has an angle of 53.13°. Thus, θ_1 = 53.13° - 30° = 23.13° and P_1= (208)(8)cos 23.13°=1530W.

Now consider wattmeter 2, Figure 5. Since W_2 is connected to terminals *c*-*b*, the voltage across it is $\mathbf{V}_{cb}$ and the current through it is $\mathbf{I}_c$. But $\mathbf{V}_{cb} = \mathbf{V}_{bc} = 208\ \text{V}\angle 90°$ and $\mathbf{I}_c = 8\text{A}\angle 173.13°$. The angle between $\mathbf{V}_{cb}$ and $\mathbf{I}c$ is thus 173.13° - 90° = 83.13°. Therefore,

$P_2 = V_{cb}I_c\cos\theta_2 = (208)(8)\cos 83.13° = 199\ \text{W}$

and $P\text{T} = P1 + P2 = 1530 + 199 = 1729\ \text{W}$.

Note that one of the wattmeters reads lower than the other. (This is generally the case for the two-wattmeter method.)

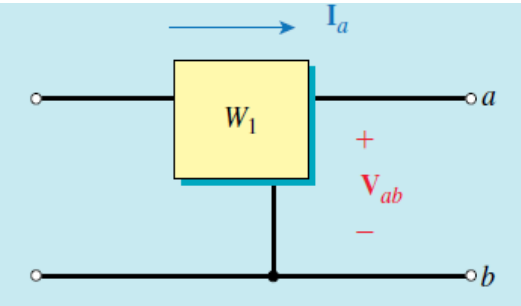


Fig.4

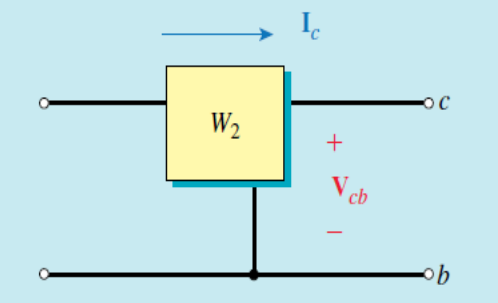
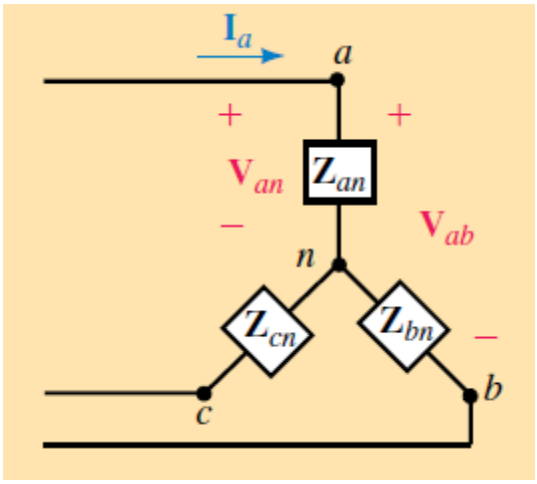


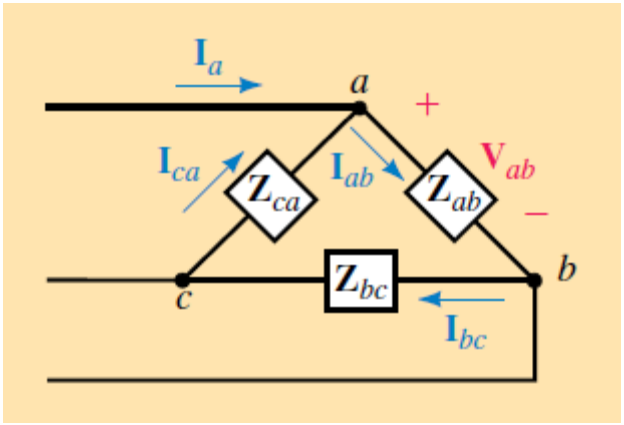
Fig.5

Table –1 summarizes the relationships developed so far. Note that in balanced systems (Y or Δ), *all* voltages and *all* currents are balanced.

TABLE –1 Summary of Relationships (Balanced System). All Voltages and Currents Are Balanced



(a) Y- connection



(b) Δ - connection

$$\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30^\circ$$
$$\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_{an}$$
$$\mathbf{Z}_{an} = \mathbf{Z}_{bn} = \mathbf{Z}_{cn}$$

$$\mathbf{I}_a = \sqrt{3}\mathbf{I}_{ab}\angle -30^\circ$$
$$\mathbf{I}_{ab} = \mathbf{V}_{ab}/\mathbf{Z}_{ab}$$
$$\mathbf{Z}_{ab} = \mathbf{Z}_{bc} = \mathbf{Z}_{ca}$$

$$\text{Generator } \mathbf{E}_{AB} = \sqrt{3}\mathbf{E}_{AN}\angle 30^\circ$$

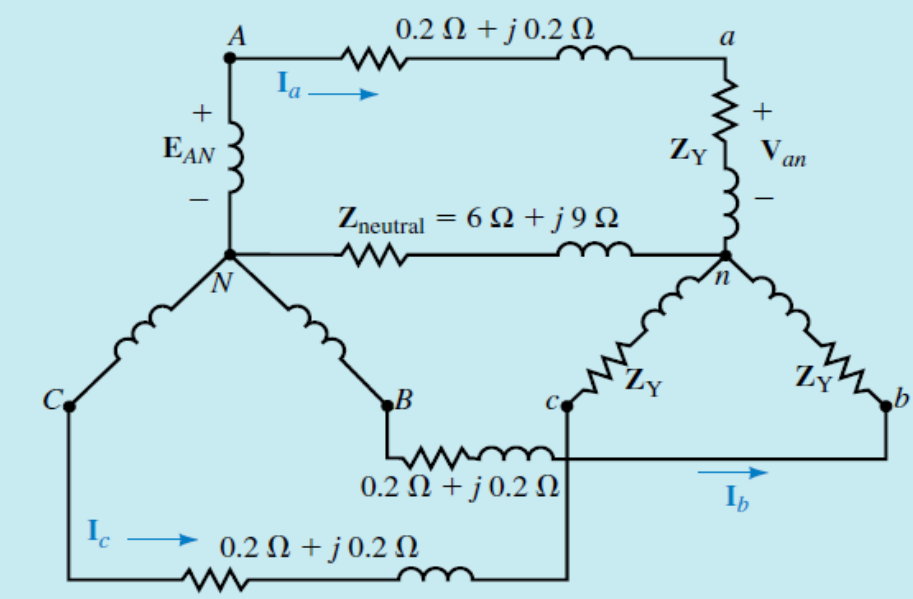
EXAMPLE 1 For Figure 1, $\mathbf{E}_{AN} = 120\ \text{V}\angle 0°$.

a. Solve for the line currents.

b. Solve for the phase voltages at the load.

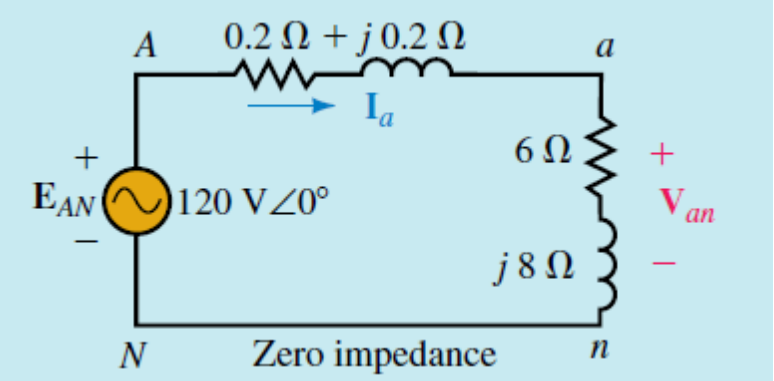
c. Solve for the line voltages at the load.

(a)



(a) $\mathbf{Z}_Y = 6\ \Omega + j\,8\ \Omega$

(b)



a. Reduce the circuit to its single-phase equivalent as shown in (b).

$$\mathbf{I}_a = \frac{\mathbf{E}_{AN}}{\mathbf{Z}_T} = \frac{120\angle 0°}{(0.2 + j0.2) + (6 + j8)} = 11.7\ \text{A}\angle -52.9°$$

Therefore,

$$\mathbf{I}_b = 11.7\ \text{A}\angle -172.9° \quad \text{and} \quad \mathbf{I}_c = 11.7\ \text{A}\angle 67.1°$$

b. $\mathbf{V}_{an} = \mathbf{I}_a \times \mathbf{Z}_{an} = (11.7\angle -52.9°)(6 + j8) = 117\ \text{V}\angle 0.23°$

Thus,

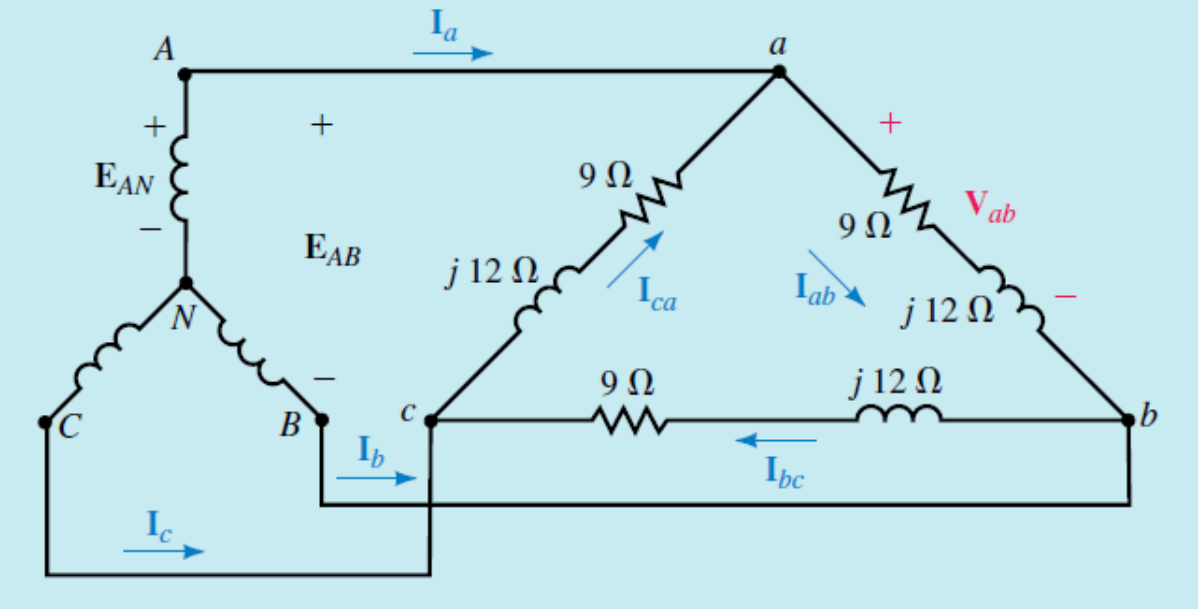
$$\mathbf{V}_{bn} = 117\ \text{V}\angle -119.77° \quad \text{and} \quad \mathbf{V}_{cn} = 117\ \text{V}\angle 120.23°$$

c. $\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30° = \sqrt{3} \times 117\angle(0.23° + 30°) = 202.6\ \text{V}\angle 30.23°$

Thus,

$$\mathbf{V}_{bc} = 202.6\ \text{V}\angle -89.77° \quad \text{and} \quad \mathbf{V}_{ca} = 202.6\ \text{V}\angle 150.23°$$

PROBLEM 2 For the circuit of Figure , **E***AB* = 208 V∠30°.
a. Determine the phase currents.
b. Determine the line currents.



Solution

a. Since this circuit has no line impedance, the load connects directly to the source and $\mathbf{V}_{ab} = \mathbf{E}_{AB} = 208\ \text{V}\angle 30°$. Current $\mathbf{I}_{ab}$ can be found as

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{208\angle 30°}{9 + j12} = \frac{208\angle 30°}{15\angle 53.1°} = 13.9\ \text{A}\angle -23.13°$$

Thus,

$$\mathbf{I}_{bc} = 13.9\ \text{A}\angle -143.13° \text{ and } \mathbf{I}_{ca} = 13.9\ \text{A}\angle 96.87°$$

b. $\mathbf{I}_a = \sqrt{3}\mathbf{I}_{ab}\angle -30° = \sqrt{3}(13.9)\angle(-30° - 23.13°) = 24\ \text{A}\angle -53.13°$

Thus,

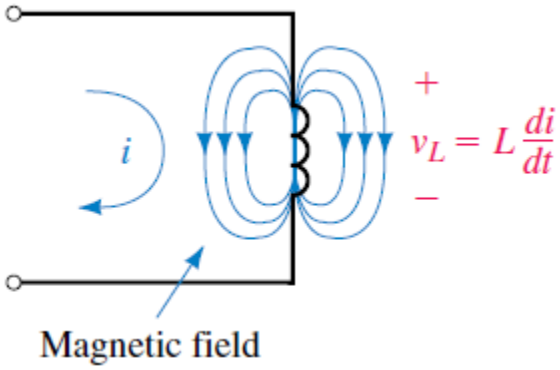
$$\mathbf{I}_b = 24\ \text{A}\angle -173.13° \quad \text{and} \quad \mathbf{I}_c = 24\ \text{A}\angle 66.87°$$

Coupled Circuits

Voltages in Air-Core Coils

To begin, consider the isolated (noncoupled) coil of Figure 1. , the voltage across this coil is given by $v_L = L\, di/dt$, where i is the current through the coil and L is its inductance. Note carefully the polarity of the voltage; the plus sign goes at the tail of the current arrow. Because the coil's voltage is created by its own current, it is called a **self-induced voltage.**

Fig.1



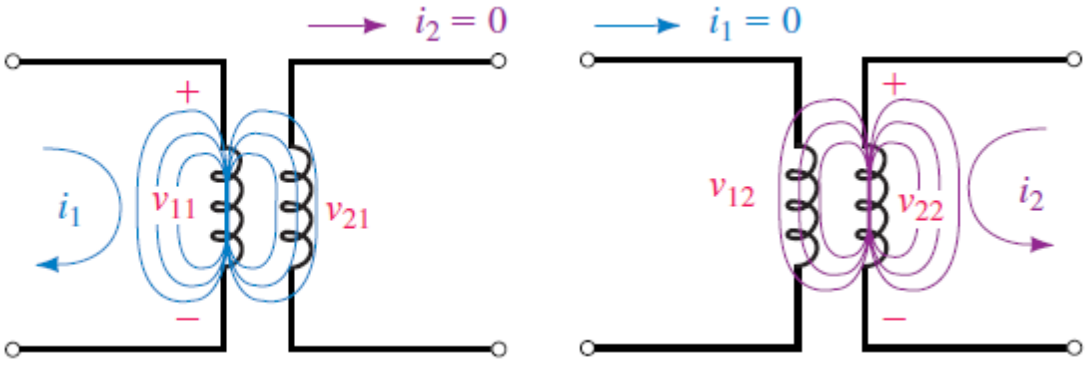
Now consider a pair of coupled coils (Figure 2). When coil 1 alone is energized as in (a), it looks just like the isolated coil of Figure 1; thus its voltage is

$$v_{11} = L_1 di_1/dt \qquad \text{(self-induced in coil 1)}$$

where L_1 is the self-inductance of coil 1 and the subscripts indicate that v_{11} is the voltage across coil 1 due to its own current. Similarly, when coil 2 alone is energized as in (b), its self-induced voltage is

$$v_{22} = L_2 di_2/dt \qquad \text{(self-induced in coil 2)}$$

For both of these self-voltages, note that the plus sign goes at the tail of their respective current arrows.



(a) (b)

Fig.2 Self and mutual voltages.

Mutual Voltages

Consider again Figure 2(a). When coil 1 is energized, some of the flux that it produces links coil 2, inducing voltage $v21$ in coil 2. Since the flux here is due to $i1$ alone, $v21$ is proportional to the rate of change of $i1$. Let the constant of proportionality be M. Then,

$$v_{21} = M di_1/dt \qquad \text{(mutually induced in coil 2)}$$

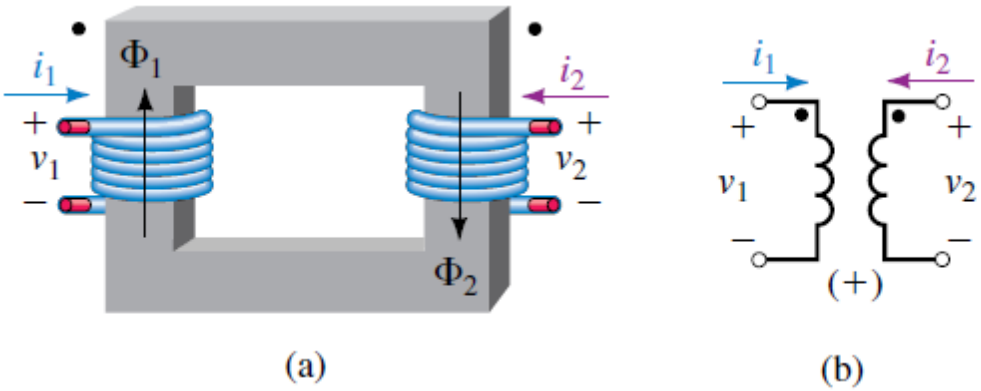
v_{21} is the **mutually induced voltage** in coil 2 and M is the **mutual inductance** between the coils. It has units of henries. Similarly, when coil 2 alone is energized as in (b), the voltage induced in coil 1 is

$$v_{12} = M di_2/dt \qquad \text{(mutually induced in coil 1)}$$

When both coils are energized, the voltage of each coil can be found by superposition; *in each coil, the induced voltage is the sum of its self-voltage plus the voltage mutually induced due to the current in the other coil.* The sign of the self term for each coil is straightforward: It is determined by placing a plus sign at the tail of the current arrow for the coil as shown in
Figures 2(a) and (b). The polarity of the mutual term, however, depends on whether the mutual voltage is additive or subtractive.

Additive and Subtractive Voltages

Whether self- and mutual voltages add or subtract depends on the direction of currents through the coils relative to their winding directions. This is best described in terms of the dot convention. Consider Figure 3(a). Comparing the coils here to Figure 4, you can see that their top ends correspond and thus can be marked with dots. Now let currents enter both coils at dotted ends. Using the right-hand rule, you can see that their fluxes add. The total flux linking coil 1 is therefore the *sum* of that produced by $i1$ and $i2$;



(a) (b)

Fig.3 When both currents enter dotted terminals, use the + sign for the mutual term in Equation(1).

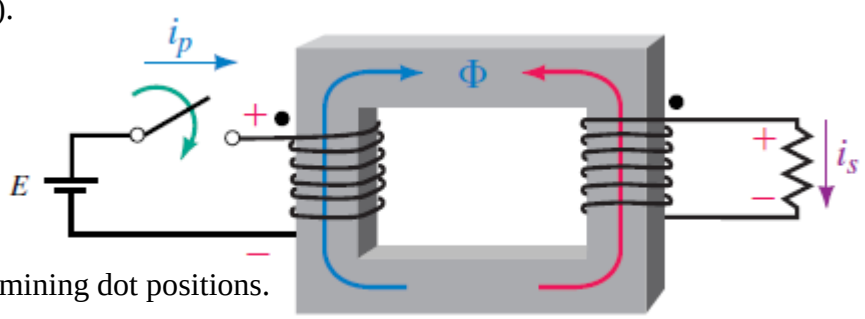


Fig.4 Determining dot positions.

therefore, the voltage across coil 1 is the sum of that produced by *i*1 and *i*2. That is,

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \qquad (1\text{ a})$$

For coil 2 :

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \qquad (1\text{b})$$

Now consider Figure 5. Here, the fluxes oppose and the flux linking each coil is the *difference* between that produced by its own current and that produced by the current of the other coil. Thus, the sign in front of the mutual voltage terms will be negative.

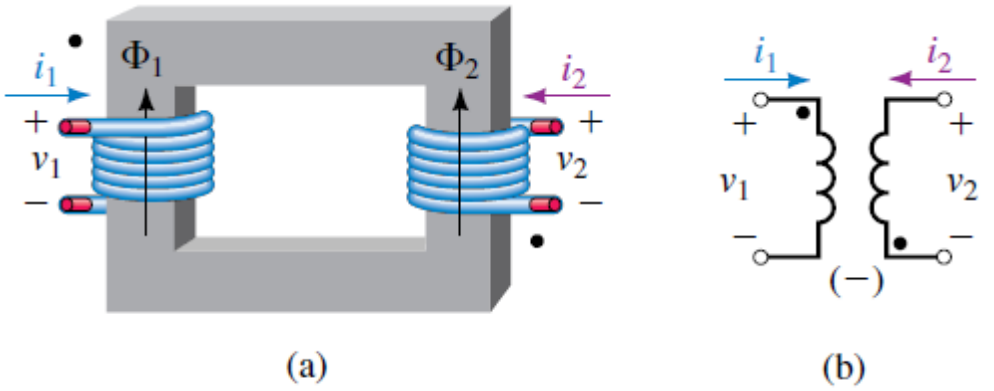
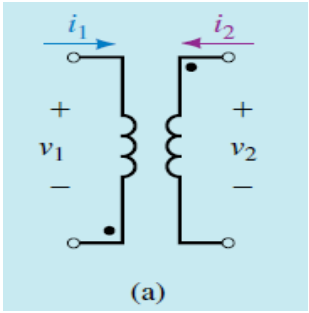


Fig.5 When one current enters a dotted terminal and the other enters an undotted terminal, use the - sign for the mutual term in Equation (1).

EXAMPLE Write equations for *v*1 and *v*2 of Figure 6 (a).

Fig.6

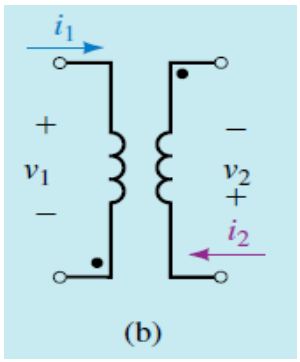


Solution Since one current enters an undotted terminal and the other enters a dotted terminal, place a minus sign in front of *M*. Thus,

$$v_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$v_2 = -M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

Write equations for *v*1



Solution

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

Coefficient of Coupling

For loosely coupled coils, not all of the flux produced by one coil links the other. To describe the degree of coupling between coils, we introduce a **coefficient of coupling,** *k*. Mathematically, *k* is defined as the ratio of the flux that links the companion coil to the total flux produced by the energized coil.

For iron-core transformers, almost all the flux is confined to the core and links both coils; thus, *k* is very close to 1. At the other extreme (i.e., isolated coils where no flux linkage occurs), $k = 0$. Thus, $0 \le k < 1$. Mutual inductance depends on *k*. It can be shown that mutual inductance, self-inductances, and the coefficient of coupling are related by the equation

$$M = k\sqrt{L_1 L_2}$$

Inductors with Mutual Coupling

If a pair of coils are in close proximity, the field of each coil couples the other, resulting in a change in the apparent inductance of each coil. To illustrate, consider Figure 7 (a), which shows a pair of inductors with selfinductances *L*1 and *L*2. If coupling occurs, the

effective coil inductances will no longer be $L1$ and $L2$. To see why, consider the voltage induced in each winding—it is the sum of the coil's own self-voltage plus the voltage mutually induced from the other coil. Since current is the same for both coils,

$$v1 = L1\ di/dt + M\ di/dt = (L1 + M)\ di/dt,\backslash$$

which means that coil 1 has an effective inductance of $L1 = L1 + M$.

Similarly, $v2 = (L2 + M)\ di/dt$, giving coil 2 an effective inductance of $L2 = L2 + M$.

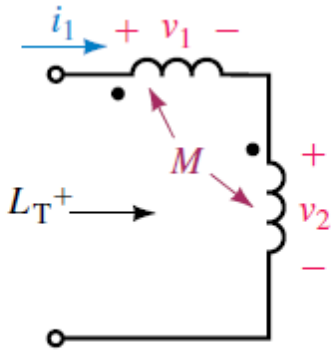
The effective inductance of the series combination [Figure 7 (b)] is then

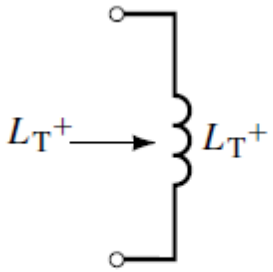
and

$$L_T^+ = L_1 + L_2 + 2M \quad \text{(henries)}$$

$$L_T^- = L_1 + L_2 - 2M \quad \text{(henries)}$$

Fig.7





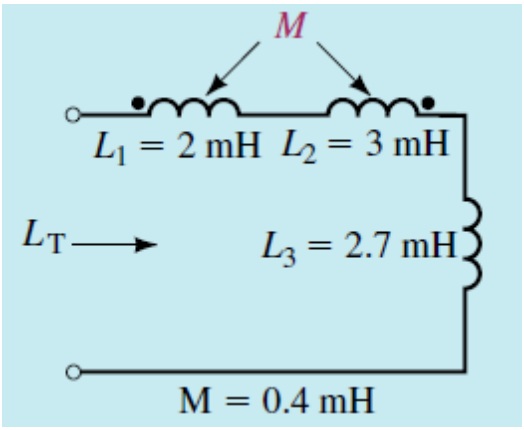
EXAM (Fig (b) $L_T^+ = L_1 + L_2 + 2\,M$

but coil (a) $L_1' = L_1 + M; L_2' = L_2 + M$

a. Determine the effective inductance of each coil.

b. Determine the total inductance of the series connection.

Fig.8



Solution

a. $L_1' = L_1 - M = 2\text{ mH} - 0.4\text{ mH} = 1.6\text{ mH}$

$L_2' = L_2 - M = 3\text{ mH} - 0.4\text{ mH} = 2.6\text{ mH}$

L_1' and L_2' are in series with L_3. Thus,

b. $L_T = 1.6\text{ mH} + 2.6\text{ mH} + 2.7\text{ mH} = 6.9\text{ mH}$